Stochastic spatial models for actin monomers-polymers dynamics

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Introduction

The mechanical properties of animals cells, such as movement or division, are mostly governed by the movement of their cortex (their surface). This cortex is filled with many proteins, that form a cytoskeleton. The most abundant protein in this cytoskeleton is actin. Actin is a protein that can aggregate to form polymers. During cell division, a dense belt of actin polymers is formed at the equator, that then shrinks in order to form two cells. In this report, we study the dynamic of actin polymers and actin single proteins

(or monomers) during this tranformation. In some models, we also study the role played by an accessory protein, the profilin. Profilin molecules can attach to actin monomers, creating an actin/profilin complex (or profilactin). This complex has different properties than actin monomers, creating different dynamics. We refer to the Introduction of the thesis of Anne Van Gorp [11] and Julien Berro [1] for a introduction to the role of actin in the cell (from mathematicans' points of view).

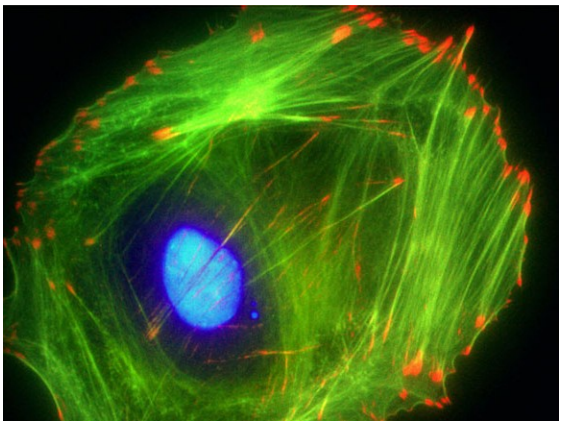

Figure 1: Visualisation of different types of molecules in the cell cortex by fluorescence. The actin polymers are visible as green filaments. Figure copied from the website [4].

Mathematical modelling of actin monomers-polymers dynamic was already studied in the thesis of Anne Van Gorp [11], but without a spatial structure. This model is relevant under the assumption that the density of monomers and polymers are homogeneous in the whole cell. This is however not the case during cell division, and that is why, in this report, we try to generalize this model to include a spatial structure. In this process, we take inspiration of works on spatially structured population dynamic, see for example [3]. We heavily rely on the framework developped by Lea Popovic and Amandine Veber in [7] to define microscopic, stochastic dynamic of actin monomers and polymers, and large population limits of those.

The main question we try to answer is the following: how does the concentration of polymers in a specific region of the spatial domain impact the concentration of monomers in the whole domain? Several aspects of this question are studied. Each polymer has a given length; does the distribution of these lengths have an impact on the concentration of monomers? Is it possible that, despite the heterogeneous distribution of the polymers, the distribution of monomers is still homogeneous? From a modelling point of view, we also need an interaction radius. How does this radius impact the global dynamic?

In Section 1, we describe the framework built in [7] to define the dynamic of particles of different species evolving in a compact space. These particles typically move as independent Brownian motions, and interact through spontaneous reactions, creating new particles and removing old ones, with some probability when they are close enough.

In Section 2, we use this framework to describe the dynamic of monomers and polymers on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Polymers in the cellular cortex are considered to be on a "belt", which in the one-dimensional analog means they are fixed at a given point, that we call 0. Monomers move around freely. In this model, we track the spatial distribution of monomers, as well as the distribution in lengths of the polymers. Tracking the length of the polymers is needed for a full description, because polymers of length zero cannot depolymerise. We show that for several orders of magnitude of the reactions' rates, these densities converge to a deterministic limit, which satisfy a PDE (40) with a peculiar behaviour, that we write and analyse.

In Section 3, we use the framework again to try to understand which features of the model create spatial disparities, at least at equilibirium. In simple model, although the polymers are fixed at zero, they

do not create imbalances in the spatial distribution of monomers at stationarity. We manage to create spatial disparities by including a second type of monomers.

In Section 4, we investigate the properties of the microscopic model in a specific very simple case. We show a kind of homogeneity principle: if the number of particles is of order of magnitude N, then there is typically $2\varepsilon N$ particles in the ball of radius ε (of volume 2ε), for any $\varepsilon > 0$. We give a way to use this idea to recover a result about the local time of Brownian motion.

Sections 2, 3 and 4 can be read almost independently, but they all rely on Section 1.

Notation and basic definitions

- We use $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$.
- We define the one-dimensional torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, which we equip with the standard metric $d(y, y') := \inf\{|\tilde{y} - \tilde{y}'| \mid \tilde{y} \in y + \mathbb{Z}, \tilde{y}' \in y' + \mathbb{Z}\}$. This torus has an origin $0 \in \mathbb{T}$.
- We use $\varepsilon \in]0, \frac{1}{2}[$ for the reaction radius. We assume that it is small compared to the size of the domain, i.e., $\varepsilon \ll 1$.
- A *proximity kernel* (with radius ε) is a function $\Gamma_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ which has support in $[-\varepsilon, \varepsilon]$, and such that $\int_{\mathbb{R}} \Gamma_\varepsilon(y)dy = 1$. We also assume that the proximity kernel is symetric, i.e., $\Gamma_\varepsilon(-y) = \Gamma_\varepsilon(y)$ for all $y \in \mathbb{R}$.
- We define two functions $\mathrm{Id}, \mathrm{Sq} : \mathbb{R} \to \mathbb{R}$ by $\mathrm{Id}(x) = x$ and $\mathrm{Sq}(x) = x^2$.
- If E is a metric space and I is an interval, a function $f : I \to E$ is called *càdlàg* if it has left limits and is right continuous everywhere on I. For a real càdlàg function $f : I \to \mathbb{R}^d$, define its jump function $\Delta f : I \to \mathbb{R}^d$ by $\Delta f(t) = f(t) - f(t-)$. The space of all càdlàg functions from I to E is denoted by $\mathbb{D}(I, E)$.
- We also define $\mathcal{M}(E)$ as the set of all finite measures on E, and $\mathcal{M}_p(E)$ as the set of all finite counting measures on E, i.e., finite sums of Dirac masses. Given a measure $\mu \in \mathcal{M}(E)$ and a measurable function $f : E \to \mathbb{R}$, we denote the integral of f with respect to μ by $\langle \mu, f \rangle := \int_E f d\mu$. Given a point measure $\mu \in \mathcal{M}_p(E)$, the measure describing the sampling of $n \geq 0$ "particles" without replacement will be denoted by

$$\mu^{\otimes \downarrow n}(dp_1, \dots, dp_n) := \mu(dp_1)(\mu - \delta_{p_1})(dp_2) \cdots (\mu - \delta_{p_1} - \cdots - \delta_{p_{n-1}})(dp_n). \tag{1}$$

  By convention, $\mu^{\otimes \downarrow n}$ is the null measure if $n = 0$ or $n > \langle \mu, 1 \rangle$.
- A *Polish space* is a metrisable, separable and complete topological space.
- If $(B_t)_{t \geq 0}$ denotes standard Brownian motion, a Brownian motion with diffusion constant $D > 0$ is a process that has the law of $X := (\sqrt{2D} B_t)_{t \geq 0}$, so that $\mathrm{Var}(X_t) = 2Dt$. The factor 2 is there so that the Fokker-Plank equation for the density $p(t, \cdot)$ of X_t satisfies $\partial_t p = D\, \partial_{xx} p$.

We refer to the appendix for a small recap on (discontinuous) stochastic calculus, and the Skorokhod topology on càdlàg paths.

1 A framework for the stochastic modelling of reaction-diffusion processes

The following framework is taken from [7], in a simplified version. Even in this lighter form, it is more general than just monomers and polymers. It describes what is called *reaction-diffusion* systems: particles

from a finite number of species evolves in a given spatial domain, evolving independently for most of the time, and interacting with each other at random times, through chemical reactions. It defines a random dynamic of a pool of particles.

1.1 Informal definition of the process

Consider $n \geq 1$ species of particles, that evolve in a compact geographical domain $E \subseteq \mathbb{R}^d, d \geq 1$, with smooth boundary (say, $\mathcal{C}^3$). Denote the different species by $A_1, \ldots, A_n$. A spatial configuration of particles is represented by a finite point measure on the state space $\mathcal{P} := \{A_1, \ldots, A_n\} \times E$. For example, the point measure

$$\delta_{(A_1,y_1)} + 2\delta_{(A_2,y_2)} + \delta_{(A_2,y_3)} + \delta_{(A_4,y_1)}$$

represents the configuration where one particle of species A_1 is situated at the point $y_1 \in E$, two particles of species A_2 are at the point y_2, one particle of species A_2 is at the point y_3, and one particle of species A_4 is at the point y_1. A point $p \in \mathcal{P}$ will implicitly be denoted by $p = (x, y)$ (and $p_i = (x_i, y_i)$, etc.).

We now define how these configurations should evolves through time. There are two types of evolution: reactions and diffusion.

Diffusion: Each species A_i is associated with a diffusion constant $D_i \geq 0$, and until it reacts, a molecule of species A_i evolves according to Brownian motion with diffusion factor D_i (i.e., the variance at time t is $2D_it$), normally reflected at the boundary of the domain E. Note that in the following, we will take $E = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ to be the circle, so there will be no reflection at the boundary.

Reactions: Consider a set of reactions R, with products and reactants being species in $A_1, \ldots, A_n$. A reaction $r \in R$ is described by its stoichiometric coefficients $\nu_{r,i}, \nu'_{r,i} \in \mathbb{Z}_+$, for $1 \leq i \leq n$. $\nu_{r,i}$ corresponds to the number of reactants of species A_i in reaction r, and $\nu'_{r,i}$ corresponds to the number of products of species A_i in reaction r. For example, the reaction

$$A_1 + 2A_2 \mapsto 2A_1 + A_3$$

will have stoichiometric coefficients $\nu_{r,1} = 1, \nu_{r,2} = 2, \nu_{r,3} = 0, \nu'_{r,1} = 2, \nu'_{r,2} = 0, \nu'_{r,3} = 1$. Reactions are divided into two categories: localised and non-localised reactions. A localised reaction r can only happen at a fixed point $y_r \in E$, while a non-local reaction r can happen everywhere in E. We denote the set of non-localised reactions by R_{NL}, and the set of localised reactions by R_{L}, so that $R = R_{\mathrm{NL}} \sqcup R_{\mathrm{L}}$. We define $\rho_r := \delta_{y_r}$ for a localised reaction $r \in R_{\mathrm{L}}$, and $\rho_r := \mathrm{Leb}$ the Lebesgue measure on E for a non-localised reaction $r \in R_{\mathrm{NL}}$.

Every reaction has a given rate $\lambda_r > 0$, which is a general factor describing how fast the reaction should happen. We also consider a global proximity kernel Γ_ε, of radius $\varepsilon > 0$. Recall that this radius represents the reaction radius, and is typically small in front of the size of E.

Let us focus on a reaction that we write

$$r : A_1^r + \cdots + A_{k_r}^r \mapsto B_1^r + \cdots + B_{k'_r}^r, \tag{2}$$

where repetitions are allowed (e.g. $A_1^r = A_2^r$). Between times t and $t + dt$, given that the spatial configuration of species is given at time t by $\mu_t \in \mathcal{M}(\mathcal{P})$, reaction r happens in the time interval $[t, t + dt]$ at location $\overline{y} \in E$, between a given ordered set of particles $p_1, \ldots, p_{k_r}$ with probability

$$\lambda_r \rho_r(d\overline{y}) \Big(\prod_{i=1}^{k_r} \mathbb{1}_{x_i = A_i^r} \Gamma_\varepsilon(y_i - \overline{y}) \Big) dt. \tag{3}$$

We sample the reactants without replacement, meaning that reaction r happens in the time interval $[t, t + dt]$ at location $\overline{y} \in E$ with probability

$$\lambda_r \rho_r(d\overline{y}) \int_{\mathcal{P}^{k_r}} \mu_t^{\otimes \downarrow k_r}(dp_1, \ldots, dp_{k_r}) \Big(\prod_{i=1}^{k_r} \mathbb{1}_{x_i = A_i^r} \Gamma_\varepsilon(y_i - \overline{y}) \Big) dt. \tag{4}$$

Therefore, integrating over the possible reaction locations, reaction r happens in the time interval $[t, t+dt]$ with probability

$$\Lambda_r(\mu_t)dt := \lambda_r \int_E \int_{\mathcal{P}^{k_r}} \rho_r(d\overline{y}) \mu_t^{\otimes\downarrow k_r}(dp_1, \ldots, dp_{k_r}) \Big(\prod_{i=1}^{k_r} \mathbb{1}_{x_i = A_i^r} \Gamma_\varepsilon(y_i - \overline{y}) \Big) dt, \tag{5}$$

and the probability that some reaction happens between times t and $t+dt$ is

$$\sum_r \Lambda_r(\mu_t)dt = \sum_{r \in R} \left(\lambda_r \int_E \int_{\mathcal{P}^{k_r}} \rho_r(d\overline{y}) \mu_t^{\otimes\downarrow k_r}(dp_1, \ldots, dp_{k_r}) \Big(\prod_{i=1}^{k_r} \mathbb{1}_{x_i = A_i^r} \Gamma_\varepsilon(y_i - \overline{y}) \Big) \right) dt. \tag{6}$$

The resulting process will be Markovian, so that the probability of a reaction between times t and $t+dt$ is independent of what happened before time t, conditionally on the state of the system at time t. The construction will be made more formal in the following paragraphs, with the help of exponential clocks.

When the reaction happens at time t and location $\overline{y}$, the reactants $p_1, \ldots, p_{k_r}$ are removed, and the products $B_1^r, \ldots, B_{k_r'}^r$ are created at location $\overline{y}$. In other words, if μ_{t-} is the state of the system before the reaction, the state of the system after the reaction is

$$\mu_t = \mu_{t-} - \sum_{i=1}^{k_r} \delta_{p_i} + \sum_{j=1}^{k_r'} \delta_{(B_j^r, \overline{y})}. \tag{7}$$

Remark 1.1. It is also possible to slightly change the location of creation of the product. For example, in a non-localised reaction r of type

$$A \mapsto B,$$

it can be natural to create particle B at the location where A is removed. In that case, we replace (3) by

$$\lambda_r \rho_r(d\overline{y}) \ \mathbb{1}_{x_1 = A} \mathbb{1}_{y_i = \overline{y}} \ dt, \tag{8}$$

and the following equations (4), (5) and (6) accordingly. When the reaction happens we apply

$$\mu_t = \mu_{t-} - \delta_{(A, \overline{y})} + \delta_{(B, \overline{y})}.$$

Similarly, if the reaction r is of the type

$$A + B \mapsto A + C,$$

then it can be natural not to move particle A during the reaction (in particular if particle A is a very heavy particle, or a polymer). In that case, we replace (3) by

$$\lambda_r \rho_r(d\overline{y}) \ \Big(\mathbb{1}_{x_1 = A} \mathbb{1}_{x_2 = B} \Gamma_\varepsilon(y_2 - \overline{y}) \Big) dt. \tag{9}$$

and the following equations (4), (5) and (6) accordingly. Moreover when the reaction happens we apply

$$\mu_t = \mu_{t-} - \delta_{(B, y_2)} + \delta_{(C, y_2)},$$

not modifying the counting measure of particles of type A.

1.2 Construction of the process and main properties

We now give a description of the construction of the process we just described. The reader only seeking an informal description of the system can skip until Proposition 1.6. The following construction comes from Section 2.2 in [7].

Definition 1.2. Recall that we have chosen

- A geographical compact space E,
- A set of species $A_1, \dots, A_n$,
- Diffusion constants $D_1, \dots, D_n \geq 0$,
- A set of localised and non-localised reactions $R = R_{\mathrm{L}} \sqcup R_{\mathrm{NL}}$, each characterised by its stoichiometric coefficients $\nu_{r,i}, \nu'_{r,i} \in \mathbb{Z}_+, 1 \leq i \leq n$, this reaction has $k_r := \sum_i \nu_{r,i}$ reactants and $k'_r := \sum_i \nu'_{r,i}$ products,
- For every local reaction $r \in \mathbb{R}_{\mathrm{L}}$, a location $\overline{y}_r \in E$,
- Possible choices of relocation of every non-localised reaction $r \in R_{\mathrm{NL}}$ as in Remark 1.1,
- A reaction radius $\varepsilon > 0$, and proximity kernel Γ_ε, which will be the same for all reactions for simplicity.

Given these, and an initial measure $\mu_{(0)} \in \mathcal{M}_p(\mathcal{P})$, we construct a measure-valued process $\mu \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}(\mathcal{P}))$ as follows.

1. Start with $\mu_0 = \mu_{(0)}$, and write
$$\mu_0 = \sum_{i \in I^0} \delta_{(x_i, y_i)},$$
for some index set I^0.

2. Define $\overline{\mu}^0_t$ for $t \geq 0$ by
$$\overline{\mu}^0_t = \sum_{i \in I^0} \delta_{(x_i, Y^0_i(t))}, \tag{10}$$
where $(Y^0_i(t))_{i \in I^0}$ are independent Brownian motions on the torus, originated at $(y_i)_{i \in I^0}$, where $Y^0_i(t)$ has diffusion coefficient D_j if $x_i = A_j$. Note that if $D_j = 0$, then the particle does not move.

3. Let $(\mathcal{E}_r)_{r \in R}$ be independent exponentially distributed random variables, also independent of the previous construction. Define, for $r \in R$,
$$\tau^1_r := \inf\left\{ s > 0 : \int_0^s \Lambda_r(\overline{\mu}^0_u) du > \mathcal{E}_r \right\},$$
where Λ_r is defined in (5). The variables τ^1_r are a.s. positive, and have a density with respect to Lebesgue measure, so they are a.s. distinct. Define
$$\tau^1 = \min_{r \in R} \tau^1_r,$$
$$r^1 = \operatorname*{argmin}_{r \in R} \tau^1_r,$$
which will be the time and index of the first reaction. Set $\mu_t := \overline{\mu}^0_t$ for $0 \leq t < \tau^1$. Sample $\overline{y} \in E$, $p_1, \dots, p_{k_r} \in \mathcal{P}$ according to the density
$$\frac{\lambda_{r^1}}{\Lambda_{r^1}(\mu^0_t)} \rho_{r^1}(d\overline{y}) \mu_t^{\otimes \downarrow k_{r^1}}(dp_1, \dots, dp_{k_{r^1}}) \Big(\prod_{i=1}^{k_{r^1}} \mathbb{1}_{x_i = A^r_i} \Gamma_\varepsilon(y_i - \overline{y}) \Big), \tag{11}$$

and using the notation (2) for reaction r^1, set

$$\mu_{\tau^1} := \overline{\mu}^0_{\tau^1} - \sum_{i=1}^{k_{r^1}} \delta_{p_i} + \sum_{j=1}^{k'_{r^1}} \delta_{(B_j^{r^1}, \overline{y})}. \tag{12}$$

Again, some different choices can be made here in line with Remark 1.1. We now have defined $(\mu_t)_{0\le t\le \tau^1}$. Note that we always have $\mu_{\tau^1} \in \mathcal{M}_p(\mathcal{P})$.

4. Start again the first three steps, starting at time τ^1, with the measure μ_{τ^1}. Write $\mu_{\tau^1} = \sum_{i\in I_1} \delta_{(x_i,y_i)}$, define for $t \ge \tau^1$, $\overline{\mu}^1_t = \sum_{i\in I_1} \delta_{(x_i, Y_i^1(t))}$, where $Y_i^1(t)$ are independent Brownian motions with the right diffusion constants. Pick new exponential variables $\mathcal{E}^1_r, r \in R$ of parameter one, and use them to choose the time $\tau^2 > \tau^1$ of the next reaction, and r^2 the index of the second reaction. Sample $\overline{y}, p_1, \dots, p_{r^2}$ similarly, and define $\mu_t = \overline{\mu}^1_t$ for $\tau^1 \le t < \tau^2$, and $\mu_{\tau^2} = \overline{\mu}^1_{\tau^2} - \sum_{i=1}^{k_{r^2}} \delta_{p_i} + \sum_{j=1}^{k'_{r^2}} \delta_{(B_j^{r^2}, \overline{y})}$.

5. Iterate again, defining μ_t for $0 \le t < \lim_{j\to\infty} \tau^j =: \tau^\infty$.

Denote the random process μ constructed this way by

$$M(E, \{A_1, \dots, A_n\}, \{D_1, \dots, D_n\}, R_{\mathrm{L}}, R_{\mathrm{NL}}, \{\lambda_r\}_{r\in R}, \Gamma_\varepsilon, \mu_{(0)}), \tag{13}$$

where any additional information (e.g. the localisation y_r of a localised reaction r) is implicit.

Remark 1.3. We point out that showing that $\tau^\infty = \infty$ a.s. requires a case by case analysis. If all reactions have at most one reactant, it will always be the case, this can be shown using the fact that the total reaction rate is bounded by a constant times the total mass of the system.

The resulting process $(\mu_t)_{t\ge 0}$ is a continuous-time Markov process on $\mathcal{M}(\mathcal{P})$. We define its infinitesimal generator as the sum of the infinitesimal generators for diffusion and for the reactions. Before defining these infinitesimal generators, we need an appropriate set $\mathcal{F}$ of test functions.

Definition 1.4. • Define, for any subset Ω of $\mathbb{R}^d$, $\mathcal{C}^2_b(\Omega) := \{f \in \mathcal{C}^2(\Omega) : f, f', f'' \text{ are bounded}\}$. For $\mu \in \mathcal{M}(\mathcal{P})$, $F : \mathbb{R} \to \mathbb{R}$ and $f : P \to \mathbb{R}$, we write

$$F_f(\mu) = F(\langle \mu, f\rangle), \tag{14}$$

whenever it is defined.

- For $y \in \partial E$, define $n(y) \in \mathbb{R}^d$ to be the outward vector normal to the boundary of E at y. Define $\mathcal{C}^{2,\perp}(E) := \{f \in \mathcal{C}^2(E) \,|\, \forall y \in \partial E, f'(y)\cdot n(y) = 0\}$, and define $\mathcal{C}^{2,\perp}(\mathcal{P}) \subseteq \mathcal{C}(\mathcal{P})$ to be the set of functions $f : \mathcal{P} \to \mathbb{R}$ such that for any species $i \in \{1, \dots, n\}$, the corresponding marginal $f_i : E \to \mathbb{R}$ belongs to $\mathcal{C}^{2,\perp}(E)$.

  We note that the set $\{\langle \mu, f\rangle, f \in \mathcal{C}^{2,\perp}(\mathcal{P})\}$ characterizes a measure $\mu \in \mathcal{M}(\mathcal{P})$.

- Define a set $\mathcal{F}$ of test functions on $\mathcal{M}(\mathcal{P})$ by

$$\mathcal{F} = \{F_f \mid F \in \mathcal{C}^2_b(\mathbb{R}), f \in \mathcal{C}^{2,\perp}(\mathcal{P})\}, \tag{15}$$

- Define the respective infinitesimal generators for the diffusions to be, for $F_f \in \mathcal{F}$ and $\mu \in \mathcal{M}(\mathcal{P})$,

$$\mathcal{D}F_f(\mu) = \sum_{i=1}^n D_i \Big(F'(\langle \mu, f\rangle)\langle \mu, \Delta_y f_i\rangle + F''(\langle \mu, f\rangle)\langle \mu^M, |\nabla_y f_i|^2\rangle \Big). \tag{16}$$

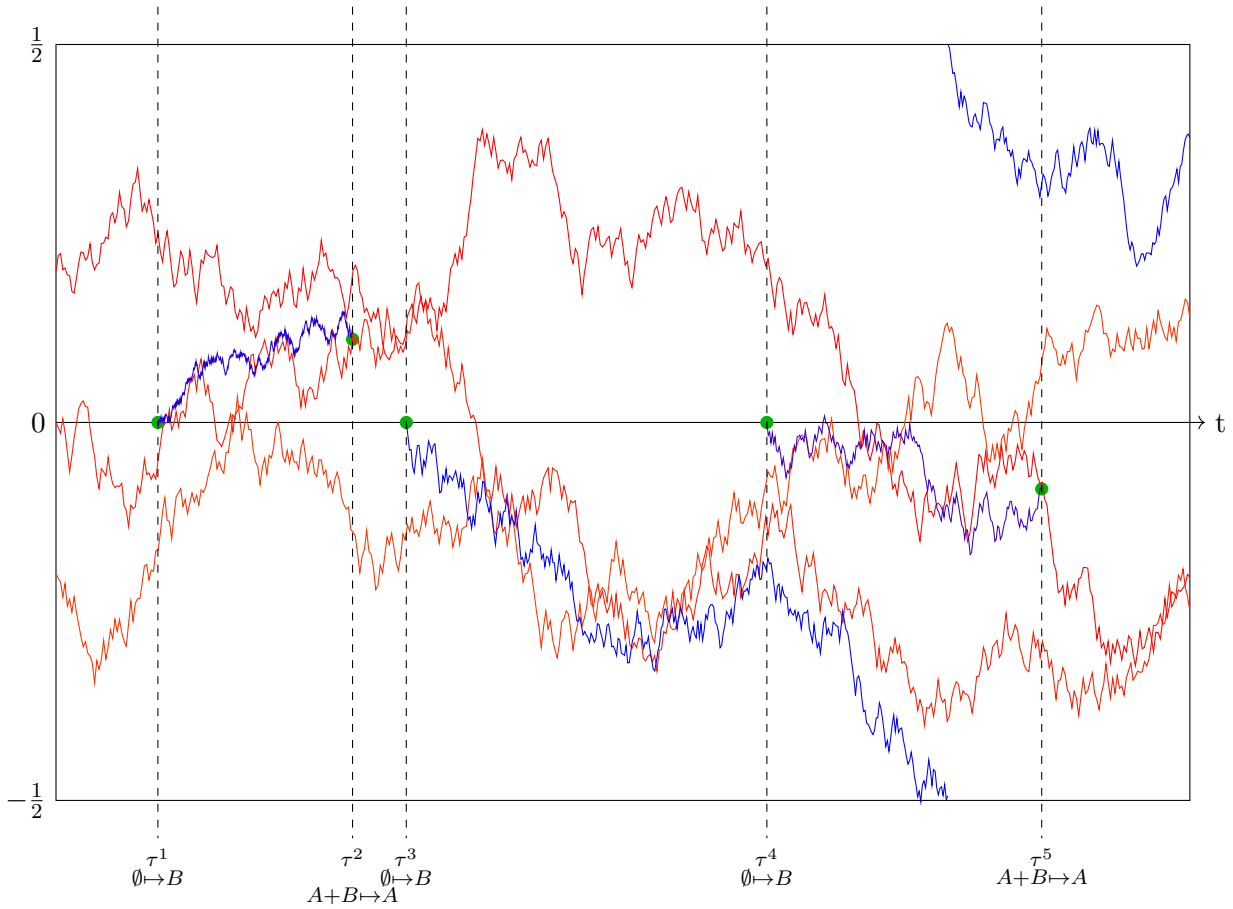

Figure 2: Exemple of a dynamic in $\mathbb{T}$ with two reactions $\emptyset \mapsto B$ and $A+B \mapsto A$. A particles are red(-ish) and B particles are blue(-ish). Reactions are indicated by green dots.

Define the respective infinitesimal generators of the reactions $r \in R$ given by (2) to be, for $F_f \in \mathcal{F}$ and $\mu \in \mathcal{M}(\mathcal{P})$,

$$G_r F_f(\mu) = \lambda_r \int_E \int_{\mathcal{P}^{k_r}} \rho_r(d\overline{y}) \mu_t^{\otimes \downarrow k_r}(dp_1, \dots, dp_{k_r}) \Big(\prod_{i=1}^{k_r} \mathbb{1}_{x_i = A_i^r} \Gamma_\varepsilon(y_i - \overline{y}) \Big) \Big(F_f(\mu - \sum_{i=1}^{k_r} \delta_{p_i} + \sum_{j=1}^{k'_r} \delta_{(B_j^r, \overline{y})}) - F_f(\mu) \Big). \tag{17}$$

The infinitesimal generator L of the reaction-diffusion process is then the sum of all the above operators

$$L F_f(\mu) = \mathcal{D} F_f(\mu) + \sum_{r \in R} G_r F_f(\mu). \tag{18}$$

For the moment it is not clear why this defines a Markov process. We will later construct a Markov process that has L as its infinitesimal generator.

- We say that a (random) process $\mu \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}(\mathcal{P}))$ satisfies the *martingale problem associated to* L if for every $F_f \in \mathcal{F}$, the process

$$\Big(F_f(\mu_t) - F_f(\mu_0) - \int_0^t L F_f(\mu_s) ds \Big)_{t \geq 0} \tag{19}$$

is a martingale.

Definition 1.5. Define the following hypothesis of control over the total mass of particles:

(A1) For every $T > 0$, μ is a.s defined up to time T and we have

$$\sup_{0\le t\le T} \langle \mu_t, 1\rangle < \infty, \qquad \sup_{0\le t\le T} \mathbb{E}[\langle \mu_t, 1\rangle^{1+\max_r k_r}] < \infty.$$

Proposition 1.6. *Assume Hypothesis* (A1) *holds true. Then, the process* $(\mu_t)_{t\ge 0}$ *from Definition 1.2 is well defined at all time, and satisfies the martingale problem associated to* L*. Moreover, the process* (19) *is also a martingale for* $F = \mathrm{Id}$ *or* $F = \mathrm{Sq}$*, and* $f \in \mathcal{C}^{2,\perp}(\mathcal{P})$*.*

Proof. We do not give the proof here, and refer to the proof of Theorem 2.6 in [7]. □

Remark 1.7. It is possible to extend the framework quite a bit. For example, every particle $p_i = (x_i, y_i)$ can diffuse according to a diffusion process whose diffusion constant also depends on its location y_i, and could have a drift (outside the boundary). Moreover, the rates λ_r can be non-constant. We can make them depend on the location $\overline{y}$ of the reaction, and also on the total distribution of species at the time of the reaction. We won't need these extensions in this report, and we refer to the framework developped in [7], Section 2, for more details.

It is also possible to extend the model to add traits (or characteristics) to each particle. We make the following hypothesis: traits can only change by jumps (and not continuously), and if a finite number of traits are present, then a finite number of reactions can happen. In that case, it is almost the same as without traits, but with a non-countable number of species. Here is how we can construct such an extension. Let $F_1, \dots, F_n$ be Polish spaces, F_i being the trait space for species A_i. The phase space is now $\mathcal{P} = \{A_1\} \times E \times F_1 \sqcup \cdots \sqcup \{A_n\} \times E \times F_n$. At any point in the dynamic, there is only a finite number of species, and therefore a finite number of traits have appeared, and a finite number of reactions have happened. We are back to the usual setting.

2 Macroscopic description of the length of the polymers

Although it is possible to define a microscopic model to describe the dynamic of a system of interacting particles, its complexity can be high and we need relevant approximations to obtain some information about the global behaviour of the system. In this section, we develop a model for the dynamic of polymers and monomers in a one-dimensional domain, fitting the framework of the last section. We start with a large number N of monomers, and normalise the measure $\mu^N \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_p(\mathcal{P}))$ by considering that each particle has a mass of $\frac{1}{N}$, instead of 1. We show that this measure converges in law as N goes to infinity to a deterministic measure $\mu \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}(\mathcal{P}))$. This is known as a *large population limit*. We describe μ as the solution of a PDE, that we analyse.

2.1 Stochastic microscopic model

We consider a pool of monomers and polymers evolving on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The polymers do not move and are fixed at the same position $0 \in \mathbb{T}$. The monomers evolve freely in the whole space $\mathbb{T}$, each one according to an independent Brownian motion with diffusion constant $D > 0$.

We consider the trait space $\mathcal{P}_0 := \{M, S\} \times \mathbb{T} \times \mathbb{R}_+$ (M is for Monomer, S is for Simple polymer). The torus part represents the spatial location, and the real positive part represents the length of a given polymer. By convention, monomers have a length of zero. Since polymers are considered to be fixed at $0 \in \mathbb{T}$, all the mass will be on the set $\mathcal{P} := \{M\} \times \mathbb{T} \times \{0\} \sqcup \{S\} \times \{0\} \times \mathbb{R}_+ \subset \mathcal{P}_0$. We consider that before rescaling, adding a monomer to a polymer increases its length by $\delta > 0$, and that the length of every polymer belongs to $\delta\mathbb{Z}_+$.

The model consists of the two following reactions of polymerisation and depolymerisation:

$$(S, 0, l) + (M, y, 0) \to (S, 0, l + \delta) \qquad l \ge \delta \tag{20}$$

$$(S, 0, l) \to (S, 0, l - \delta) + (M, 0, 0) \qquad l \ge \delta \tag{21}$$

Note that any polymer of length zero $(S,0,0)$ is considered a dead polymer: it will never polymerise or depolymerise again.

The model fits the framework developped in Section 1, with the trait extension described in Remark 1.7. Polymer have traits in $\mathbb{R}_+$ (their length), and monomers have no traits (or the trivial trait 0).

Definition 2.1.

- Define a set $\mathcal{F}$ of test functions on $\mathcal{P}$ by

$$\mathcal{F} = \{F_f \mid F \in \mathcal{C}_b^2(\mathbb{R}), f \in \mathcal{C}^2(\{M\}\times\mathbb{T}\times\{0\}) \times \mathcal{C}_b^2(\{S\}\times\{0\}\times\mathbb{R}_+)\}, \tag{22}$$

  We abusively write $\mathcal{C}^2(\{M\}\times\mathbb{T}\times\{0\}) \times \mathcal{C}_b^2(\{S\}\times\{0\}\times\mathbb{R}_+)$ as $\mathcal{C}^2(\mathbb{T})\times\mathcal{C}_b^2(\mathbb{R}_+)$. A function $f \in \mathcal{C}^2(\mathbb{T})\times\mathcal{C}_b^2(\mathbb{R}_+)$ will be denoted as $f=(f^M,f^S)$ with $f^M \in \mathcal{C}^2(\mathbb{T})$ and $f^S \in \mathcal{C}_b^2(\mathbb{R}_+)$, and a measure $\mu \in \mathcal{M}(\mathcal{P})$ will be denoted by $\mu=(\mu^M,\mu^S)$ where $\mu^M \in \mathcal{M}(\mathbb{T})$ and $\mu^S \in \mathcal{M}(\mathbb{R}_+)$. Note that the set $\{\langle \mu, f\rangle, f \in \mathcal{C}^2(\mathbb{T})\times\mathcal{C}_b^2(\mathbb{R}_+)\}$ characterizes a measure $\mu \in \mathcal{M}(\mathcal{P})$.

- Fix $\lambda^-,\lambda^+,D>0$ and a proximity kernel Γ_ε. We will model the dynamic as a continuous-time Markov chain over $\mathcal{M}(\mathcal{P})$. Define the respective infinitesimal generators of the diffusion, the depolymerisation and the polymerisation reactions by setting, for $F_f \in \mathcal{F}$ and $\mu \in \mathcal{M}(\mathcal{P})$,

$$\begin{aligned}
\mathcal{D}F_f(\mu) &= D\ F'(\langle\mu,f\rangle)\langle\mu,\Delta_y f^M\rangle + D\ F''(\langle\mu,f\rangle)\langle\mu^M,|\nabla_y f^M|^2\rangle,\\
G^-F_f(\mu) &= \lambda^- \int_\delta^\infty d\mu^S(l)\Big(F_f(\mu+\delta_{(M,0,0)}+\delta_{(S,0,l-\delta)}-\delta_{(S,0,l)})-F_f(\mu)\Big),\\
G^+F_f(\mu) &= \lambda^+ \int_\delta^\infty \int_{\mathbb{T}} d\mu^S(l)d\mu^M(y)\Gamma_\varepsilon(y)\Big(F_f(\mu-\delta_{(M,y,0)}+\delta_{(S,0,l+\delta)}-\delta_{(S,0,l)})-F_f(\mu)\Big).
\end{aligned}$$

  The infinitesimal generator L_{len} is the sum of these three operators:

$$L_{\text{len}}F_f(\mu) = \mathcal{D}F_f(\mu) + G^-F_f(\mu) + G^+F_f(\mu). \tag{23}$$

- We say that a process $\mu \in \mathbb{D}(\mathbb{R}_+,\mathcal{M}(\mathcal{P}))$ satisfies the martingale problem $\mathrm{MP}_{\text{len}}(\lambda^-,\lambda^+,D,\Gamma_\varepsilon,\delta)$ if for every $F_f \in \mathcal{F}$, the process

$$\Big(F_f(\mu_t) - F_f(\mu_0) - \int_0^t L_{\text{len}}F_f(\mu_s)ds\Big)_{t\geq 0} \tag{24}$$

  is a martingale.

Theorem 2.2. *Given any $\mu_{(0)} \in \mathcal{M}_p(\mathcal{P})$, there exists a solution $\mu \in \mathbb{D}(\mathbb{R}_+,\mathcal{M}(\mathcal{P}))$ to the martingale problem $\mathrm{MP}_{len}(\lambda^-,\lambda^+,D,\Gamma_\varepsilon,\delta)$ with initial distribution $\mu_0=\mu_{(0)}$. Moreover, the martingale problem is satisfied for $F=\mathrm{Id}$ and $F=\mathrm{Sq}$, and we can write, for $f \in \mathcal{C}^2(\mathbb{T})\times\mathcal{C}_b^2(\mathbb{R}_+)$*

$$\begin{aligned}
\langle\mu_t,f\rangle = \langle\mu_0,f\rangle + \int_0^t \Big(&D\langle\mu_s,\Delta_y f^M\rangle + \lambda^- \int_0^\infty d\mu_s^S(l)\mathbb{1}_{l\geq\delta}(f^M(0)+f^S(l-\delta)-f^S(l))\\
&+\lambda^+\int_0^\infty\int_{\mathbb{T}} d\mu_s^S(l)d\mu_s^M(y)\mathbb{1}_{l\geq\delta}\Gamma_\varepsilon(y)(-f^M(y)+f^S(l+\delta)-f^S(l))\Big)ds + Z_t^f,
\end{aligned} \tag{25}$$

where $(Z_t^f)_{t\geq 0}$ is a martingale with predictable quadratic variation

$$\begin{aligned}
\langle Z^f\rangle_t = \int_0^t \Big(&2D\langle\mu_s,|\nabla_y f^M|^2\rangle + \lambda^-\int_0^\infty d\mu_s^S(l)\mathbb{1}_{l\geq\delta}(f^M(0)+f^S(l-\delta)-f^S(l))^2\\
&+\lambda^+\int_0^\infty\int_{\mathbb{T}} d\mu_s^S(l)d\mu_s^M(y)\mathbb{1}_{l\geq\delta}\Gamma_\varepsilon(y)(-f^M(y)+f^S(l+\delta)-f^S(l))^2\Big)ds.
\end{aligned} \tag{26}$$

Proof. We only give a rough outline of the proof here, which can be done exactly as in the general framework of [7], where we assume that the monomers are non-localised, and the polymers are localised. We only need to add a length trait for the polymers, which does not impact the proof. In fact, we can even show the Theorem without the extension with traits: if we condition on the number N of total monomers (including the ones inside polymers), which is fixed by the dynamic, then we can consider that there are $N+2$ species : 1 for monomers, and $N+1$ for polymers of length $0, \delta, \dots, N\delta$. In that case it fits completely into the scope of Theorem 2.6 in [7].

We adapt Section 2.2 in [7] in this simpler case to explain how the solution is constructed.

1. Start with $\mu_0 = \mu_{(0)}$, and write

$$\mu_0 = \sum_{i \in I_0^M} \delta_{(M, y_i, 0)} + \sum_{i \in I_0^S} \delta_{(S,0,l_i)}.$$

   We write $\hat{I}_0^S = \{i \in I_0^S, l_i \geq \delta\}$ for the set of indices of the polymers that can depolymerise, because their length is positive.

2. Define $\overline{\mu}_t$ for $t \geq 0$ by

$$\overline{\mu}_t = \sum_{i \in I_0^M} \delta_{(M, Y_i^0(t), 0)} + \sum_{i \in I_0^S} \delta_{(S,0,l_i)},$$

   where $(Y_i^0(t))_{i \in I_0^M}$ are independent Brownian motions on the torus, originated at $(y_i)_{i \in I_0^M}$, with diffusion coefficient D.

3. Let $\mathcal{E}_+, \mathcal{E}_-$ be independent exponentially distributed random variables, also independent of the previous contruction. Define

$$\tau_-^1 := \inf\left\{ s > 0 : \lambda^- s |\hat{I}_0^S| > \mathcal{E}_- \right\},$$
$$\tau_+^1 := \inf\left\{ s > 0 : \lambda^+ \int_0^s \int_{\mathbb{T}} \Gamma_\varepsilon(y) d\mu_s^M(y) |\hat{I}_0^S| ds > \mathcal{E}_+ \right\}.$$

   Since $\left| \int_{\mathbb{T}} \Gamma_\varepsilon(y) d\mu_s^M(y) |\hat{I}_0^S| \right| \leq ||\Gamma_\varepsilon||_\infty |I_0^M| |\hat{I}_0^S|$, the variables $\tau_\pm^1$ are a.s. positive, and have a density with respect to the Lebesgue measure, so are a.s. distinct. Define

$$\tau^1 = \min(\tau_-^1, \tau_+^1),$$

   which will be the time of the first reaction. Set $\mu_t := \overline{\mu}_t$ for $0 \leq t < \tau^1$. If $\tau^1 = \tau_-^1$, then a depolymerisation reaction happens at time τ^1. Sample a polymer $J \in \hat{I}_0^S$ uniformly at random and define

$$\mu_{\tau^1} := \overline{\mu}_{\tau^1} + \delta_{(M,0,0)} + \delta_{(S,0,l_J - \delta)} - \delta_{(S,0,l_J)}.$$

   If $\tau^1 = \tau_-^1$, then a polymerisation reaction happens at time τ^1. Sample a polymer $J^S \in \hat{I}_0^S$ uniformly at random and a monomer $J^M \in I_0^M$ according to the (non-normalised) weights $w_i = \Gamma_\varepsilon(Y_i^0(\tau^1)), i \in I_0^M$, and define

$$\mu_{\tau^1} := \overline{\mu}_{\tau^1} - \delta_{(M, Y_{J^M}^0(\tau^1), 0)} + \delta_{(S,0,l_{J^S} + \delta)} - \delta_{(S,0,l_{J^S})}.$$

   Note that in both cases, μ_{τ^1} remains a positive measure.

4. Start again the first three steps at time τ^1, with the measure μ_{τ^1}, and iterate. We obtain sets $I_n^M, I_n^S, \hat{I}_n^S$ of monomers and polymers at each step, and times $0 < \tau^1 < \tau^2 < \cdots$ where reaction happens.

We now show by hand that $\tau^n \to \infty$ a.s. as $n \to \infty$, so that the process $(\mu_t)_{t\geq 0}$ is well-defined for all times.

For any $s \geq 0$ and $n \geq 0$, we have $\left|\int_{\mathbb{T}} \Gamma_\varepsilon(y) d\mu_s^M(y) |I_n^S|\right| \leq ||\Gamma_\varepsilon||_\infty |I_n^M||I_n^S| \leq C$ where C is a (random) constant depending only on the initial distribution $\mu_{(0)}$. Indeed, the number of polymers and the total number of monomers (including the ones inside polymers), is constant during the dynamic. Similarly for any $n \geq 1$, $|\hat{I}_n^S| \leq C'$, where C' only depends on the initial distribution. Therefore $\tau_+^{n+1} - \tau^n$ (resp. $\tau_-^{n+1} - \tau^n$) stochastically dominates a exponential variable of parameter C (resp. C'). Therefore $\tau^{n+1} - \tau^n = \min(\tau_+^{n+1} - \tau^n, \tau_-^{n+1} - \tau^n)$ stochastically dominates an a.s. positive variable $\mathcal{E}$ with law $\mathbb{P}(\mathcal{E} \leq x) = \min(2 - e^{-Cx} - e^{-C'x}, 1)\mathbb{1}_{x>0}$. Since all variables $\tau^{n+1} - \tau^n, n \geq 1$ are independent by the sequential construction, we have $\sum_{n\geq 1}(\tau^{n+1} - \tau^n) = \infty$ a.s.

We now show how to compute the predictable quadratic variation of Z_t^f. The martingale problem (24) with $F = \mathrm{Id}$ implies that Z_t^f is a martingale. The martingale problem (24) with $F = \mathrm{Sq} : a \mapsto a^2$ gives a martingale $Z_t^{f,2}$ satisfying

$$\langle \mu_t, f\rangle^2 - \langle \mu_0, f\rangle^2 = \int_0^t \Bigg(2D\langle \mu_s, f\rangle \langle \mu_s^M, \Delta f^M\rangle + 2D\langle \mu_s^M, |\nabla f^M|^2\rangle \tag{27}$$

$$+ \lambda^- \int_\delta^\infty d\mu_s^S(l) \Big(\big(f^M(0) + f^S(l+\delta) - f^S(l)\big)^2 + 2\big(f^M(0) + f^S(l+\delta) - f^S(l)\big)\langle \mu_s, f\rangle \Big) \tag{28}$$

$$+ \lambda^+ \int_\delta^\infty \int_{\mathbb{T}} d\mu_s^S(l) d\mu_s^M(y) \Gamma_\varepsilon(y) \Big(\big(- f^M(y) + f^S(l-\delta) - f^S(l)\big)^2 \tag{29}$$

$$+ 2\big(- f^M(y) + f^S(l-\delta) - f^S(l)\big)\langle \mu_s, f\rangle \Big) \Bigg) ds \quad + Z_t^{f,2}. \tag{30}$$

$$=: 2\int_0^t \langle \mu_s, f\rangle dA_s + B_t + Z_t^{f,2}, \tag{31}$$

where

$$A_t = \int_0^t \Big(D\langle \mu_s^M, \Delta f^M\rangle + \lambda^- \int_\delta^\infty d\mu_s^S(l)\big(f^M(0) + f^S(l+\delta) - f^S(l)\big)$$
$$+ \lambda^+ \int_\delta^\infty \int_{\mathbb{T}} d\mu_s^S(l) d\mu_s^M(y) \Gamma_\varepsilon(y)\big(- f^M(y) + f^S(l-\delta) - f^S(l)\big)\Big) ds,$$
$$B_t = \int_0^t \Big(2D\langle \mu_s^M, |\nabla f^M|^2\rangle + \lambda^- \int_\delta^\infty d\mu_s^S(l)\big(f^M(0) + f^S(l+\delta) - f^S(l)\big)^2$$
$$+ \lambda^+ \int_\delta^\infty \int_{\mathbb{T}} d\mu_s^S(l) d\mu_s^M(y) \Gamma_\varepsilon(y)\big(- f^M(y) + f^S(l-\delta) - f^S(l)\big)^2\Big) ds$$

are two adapted, continuous, finite variation processes. The martingale problem with $F = \mathrm{Id}$ can be written as

$$\langle \mu_t, f\rangle = \langle \mu_0, f\rangle + A_t + Z_t^f. \tag{32}$$

We write $X_t = \langle \mu_t, f\rangle$, which is a semimartingale. Therefore, by Lemma A.3 we have $[Z^f]_t = [X_\cdot - X_0]_t =$

$[X]_t - X_0^2$. We arrive at

$$\begin{aligned}
[Z^f]_t &= [X]_t - X_0^2 \\
&= X_t^2 - 2\int_0^t X_{s-} dX_s - X_0^2 && \text{by definition of the bracket} \\
&= 2\int_0^t X_s dA_s + B_t + Z_t^{f,2} - 2\int_0^t X_{s-} dX_s && \text{by equation (31)} \\
&= 2\int_0^t X_{s-} dA_s + B_t + Z_t^{f,2} - 2\int_0^t X_{s-} dX_s && \text{since } A \text{ is continuous} \\
&= 2\int_0^t X_{s-} dA_s + B_t + Z_t^{f,2} - 2\int_0^t X_{s-} dA_s + 2\int_0^t X_{s-} dZ_s^f && \text{by equation (32)} \\
&= B_t + Z_t^{f,2} + 2\int_0^t X_{s-} dZ_s^f.
\end{aligned}$$

Since $(Z_t^{f,2} + 2\int_0^t X_{s-} dZ_s^f)_{t\geq 0}$ is a martingale and B is adapted and continuous, hence predictable, and has finite variation, we identified the predictable quadratic variation of Z_t^f : $\langle Z^f \rangle_t = B_t$ as desired. □

Note that by construction, μ_t is a finite point measure for any $t \geq 0$. In other words, we have $\mu \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}_p(\mathcal{P}))$.

2.2 Large population limit

In order to understand the typical behavior of the system, we consider a sequence of solutions to the martingale problem $\mathrm{MP}_{\mathrm{len}}$ with an initial number of particles going to infinity, and normalise them to make the sequence converge to a non-trivial limit. More precisely, assume that the number of monomers grows as N, and the number of polymers grows as N'. If the total number of monomers (including the ones inside the polymers) is still of order N, then it means that the polymers have lengths of order N/N'. In order to have a continuous limit, we assume that this order of magnitude for the length goes to infinity. We therefore choose N, N' such that

$$\begin{aligned}
N &\to \infty, \\
N' &\to \infty, \\
N'' := \frac{N}{N'} &\to \infty,
\end{aligned}$$

Typical examples we would like to consider are $N' = N^\alpha$ with $0 < \alpha < 1$.

The parameters now have to be tuned properly for all reactions to happen at similar global rates.

Definition 2.3. Fix the set of parameters $\lambda^-, \lambda^+, D, \Gamma_\varepsilon$. Define μ^N to be the solution of the martingale problem $\mathrm{MP}_{\mathrm{len}}(N''\lambda^-, \frac{1}{N'}\lambda^+, D, \Gamma_\varepsilon, \frac{1}{N''})$ constructed in the proof of Theorem 2.2. Note that the length brought by a monomer to a polymer is now $\delta = \frac{1}{N''}$. Define the normalizations

$$\widetilde{\mu}^{N,M} = \frac{1}{N}\mu^{N,M}, \tag{33}$$

$$\widetilde{\mu}^{N,S} = \frac{1}{N'}\mu^{N,S}. \tag{34}$$

By Theorem 2.2, we can write

$$\langle \widetilde{\mu}_t^{N,M}, f^M\rangle = \langle \widetilde{\mu}_0^{N,M}, f^M\rangle + \int_0^t \Big(D\langle \widetilde{\mu}_s^{N,M}, \Delta_y f^M\rangle + \lambda^- \int_0^\infty d\widetilde{\mu}_s^{N,S}(l)\mathbb{1}_{l\geq \frac{1}{N''}} f^M(0) \\ - \lambda^+ \int_0^\infty \int_{\mathbb{T}} d\widetilde{\mu}_s^{N,S}(l) d\widetilde{\mu}_s^{N,M}(y)\mathbb{1}_{l\geq \frac{1}{N''}}\Gamma_\varepsilon(y) f^M(y)\Big)ds + Z_t^{N,f^M} \tag{35}$$

$$\langle \widetilde{\mu}_t^{N,S}, f^S\rangle = \langle \widetilde{\mu}_0^{N,S}, f^S\rangle + \int_0^t \Big(-\lambda^- \int_0^\infty d\widetilde{\mu}_s^{N,S}(l)\mathbb{1}_{l\geq \frac{1}{N''}} N''(f^S(l) - f^S(l-\frac{1}{N''})) \\ + \lambda^+ \int_0^\infty \int_{\mathbb{T}} d\widetilde{\mu}_s^{N,S}(l) d\widetilde{\mu}_s^{N,M}(y)\mathbb{1}_{l\geq \frac{1}{N''}}\Gamma_\varepsilon(y) N''(f^S(l+\frac{1}{N''}) - f^S(l))\Big)ds + Z_t^{N,f^S}. \tag{36}$$

If $\widetilde{\mu}^N = (\widetilde{\mu}^{N,M}, \widetilde{\mu}^{N,S})$, then for $f \in \mathcal{C}^2(\mathbb{T}) \times \mathcal{C}_b^2(\mathbb{R}_+)$ we have

$$\langle \widetilde{\mu}_t^{N}, f\rangle = \langle \widetilde{\mu}_0^{N}, f\rangle + \int_0^t \Bigg(D\langle \widetilde{\mu}_s^{N,M}, \Delta_y f^M\rangle + \lambda^- \int_0^\infty d\widetilde{\mu}_s^{N,S}(l)\mathbb{1}_{l\geq \frac{1}{N''}} \Big(f^M(0) - N''(f^S(l) - f^S(l-\frac{1}{N''}))\Big) \\ + \lambda^+ \int_0^\infty \int_{\mathbb{T}} d\widetilde{\mu}_s^{N,S}(l) d\widetilde{\mu}_s^{N,M}(y)\mathbb{1}_{l\geq \frac{1}{N''}}\Gamma_\varepsilon(y)\Big(- f^M(y) + N''(f^S(l+\frac{1}{N''}) - f^S(l))\Big)\Bigg) ds \\ + Z_t^{N,f}, \tag{37}$$

where both $(Z_t^{N,f^S})_{t\geq 0}$ and $(Z_t^{N,f^M})_{t\geq 0}$ are martingales, as well as $Z^{N,f} = Z^{N,f^M} + Z^{N,f^S}$. The predictable quadratic variation of $Z^{N,f}$ is given by

$$\langle Z^{N,f}\rangle_t = \frac{1}{N}\int_0^t \Bigg(D\langle \widetilde{\mu}_s^{N,M}, |\nabla_y f^M|^2\rangle + \lambda^- \int_0^\infty d\widetilde{\mu}_s^{N,S}(l)\mathbb{1}_{l\geq \frac{1}{N''}} \Big(f^M(0) - N''(f^S(l) - f^S(l-\frac{1}{N''}))\Big)^2 \\ + \lambda^+ \int_0^\infty \int_{\mathbb{T}} d\widetilde{\mu}_s^{N,S}(l) d\widetilde{\mu}_s^{NM}(y)\mathbb{1}_{l\geq \frac{1}{N''}}\Gamma_\varepsilon(y)\Big(f^M(y) - N''(f^S(l+\frac{1}{N''}) - f^S(l))\Big)^2\Bigg) ds. \tag{38}$$

Note that $|N''(f^S(l) - f^S(l-\frac{1}{N''}))| \leq ||\partial_l f^S(l)||_\infty$ is bounded.

The quadratic variation vanishes in the limit. It is therefore expected that the limit is a deterministic solution $\mu = (\mu^M, \mu^S)$ to the integro-differential equation given by

$$\begin{cases} \langle \mu_t^M, f^M\rangle = \langle \mu_0^M, f^M\rangle + \int_0^t \Big(D\langle \mu_s^M, \Delta_y f^M\rangle + \lambda^- \int_0^\infty d\mu_s^S(l)\mathbb{1}_{l>0} f^M(0) \\ \qquad\qquad - \lambda^+ \int_0^\infty \int_{\mathbb{T}} d\mu_s^S(l) d\mu^M(y)\mathbb{1}_{l>0}\Gamma_\varepsilon(y) f^M(y)\Big)ds, \\ \langle \mu_t^S, f^S\rangle = \langle \mu_0^S, f^S\rangle + \int_0^t \Big(-\lambda^- \int_0^\infty d\mu_s^S(l)\mathbb{1}_{l>0}\partial_l f^S(l) \\ \qquad\qquad + \lambda^+ \int_0^\infty \int_{\mathbb{T}} d\mu_s^S(l) d\mu^M(y)\mathbb{1}_{l>0}\Gamma_\varepsilon(y)\partial_l f^S(l)\Big)ds. \end{cases} \tag{39}$$

This is equivalent to the equation (40) of the next Theorem. We state some hypothesis that we use to show this convergence.

Definition 2.4. For $\mu \in \mathcal{M}(\mathcal{P})$, we write $\langle \mu, (1,l)\rangle = \langle \mu^M, 1\rangle + \langle \mu^S, l\rangle$. It corresponds to the total mass of monomers, including the ones inside polymers. We also write $\langle \mu, (1, 1+l)\rangle = \langle \mu^M, 1\rangle + \langle \mu^S, 1+l\rangle$.

Define the following hypothesis on μ_0^N:

(H1) We have a constant $C_0 < \infty$ such that a.s. $\sup_N \langle \widetilde{\mu}_0^N, (1, 1+l)\rangle \leq C_0$.

(H2) The measure $\widetilde{\mu}_0^N$ converges weakly to a deterministic measure $\mu_0 \in \mathcal{M}(\mathcal{P})$.

(H3) The support of $\widetilde{\mu}_0^N$ is contained in $\frac{1}{N''}\mathbb{Z}_+$. Note that by construction, it then remains the case a.s. for all $\widetilde{\mu}_t^N$, $t \geq 0$.

Conjecture 2.5. *Assume Hypotheses (H1)-(H2)-(H3) hold true, and fix $T > 0$. Then, the measure $\widetilde{\mu}^N$ converges weakly in $\mathbb{D}([0,T], \mathcal{M}(\mathcal{P}))$ to a deterministic measure $\mu \in \mathcal{C}([0,T], \mathcal{M}(\mathcal{P}))$, which is the only solution in this space to: for all $f \in \mathcal{C}^2(\mathbb{T}) \times \mathcal{C}_b^2(\mathbb{R}_+)$, and all $t \in [0,T]$,*

$$\begin{cases} \langle \mu_t^M, f^M \rangle = \langle \mu_0^M, f^M \rangle + \int_0^t \left(D \langle \mu_s^M, \Delta_y f^M \rangle - \left(\lambda^+ \langle \mu_s^M, \Gamma_\varepsilon f^M \rangle - \lambda^- f^M(0) \right) \langle \mu_s^S, \mathbb{1}_{l>0} \rangle \right) ds \\ \langle \mu_t^S, f^S \rangle = \langle \mu_0^S, f^S \rangle + \int_0^t \left(\left(\lambda^+ \langle \mu_s^M, \Gamma_\varepsilon \rangle - \lambda^- \right) \langle \mu_s^S, \mathbb{1}_{l>0} \partial_l f^S \rangle \right) ds. \end{cases} \tag{40}$$

More concisely, we say that μ is a weak solution to

$$\begin{cases} \partial_t \langle \mu_t^M, f^M \rangle = D \langle \mu_t^M, \Delta_y f^M \rangle - \left(\lambda^+ \langle \mu_t^M, \Gamma_\varepsilon f^M \rangle - \lambda^- f^M(0) \right) \langle \mu_t^S, \mathbb{1}_{l>0} \rangle, \\ \partial_t \langle \mu_t^S, f^S \rangle = \left(\lambda^+ \langle \mu_t^M, \Gamma_\varepsilon \rangle - \lambda^- \right) \langle \mu_t^S, \mathbb{1}_{l>0} \partial_l f^S \rangle. \end{cases} \tag{41}$$

Remark 2.6. The missing part of the proof of the conjecture is the uniqueness of the solution of the PDE. On the side of the monomers, it is relatively standard, even though the equaiton also depends on the polymers' distribution. On the side of the polymers, it is a transport equation with absorption at 0, but the velocity is not constant, and can even change sign.

The result we get without assuming uniqueness is the following:

Proposition 2.7. *Assume (H1)-(H2)-(H3), and fix $T > 0$. Then, the sequence of the laws of $(\widetilde{\mu}^N)_{N \geq 1}$ is tight in the space $\mathbb{D}([0,T], \mathcal{M}(\mathcal{P}))$. Moreover, any limit μ of a converging subsequence is deterministic, belongs to $\mathcal{C}([0,T], \mathcal{M}(\mathcal{P}))$, and is a solution in this space to: for all $f \in \mathcal{C}^2(\mathbb{T}) \times \mathcal{C}_b^2(\mathbb{R}_+)$, and all $t \in [0,T]$, we have* (40).

Remark 2.8. The same result holds true for the corresponding processes defined on $\mathbb{D}(\mathbb{R}_+, \mathcal{M}(\mathcal{P}))$. The tightness in the space $\mathbb{D}(\mathbb{R}_+, \mathcal{M}(\mathcal{P}))$ is actually equivalent to the tightness of the restricted processes in the space $\mathbb{D}([0,T], \mathcal{M}(\mathcal{P}))$ for all $T > 0$.

2.3 Proof of Proposition 2.7

We start the proof by deriving some bounds.

Lemma 2.9.

- *We have a.s. for any $t \geq 0$,*

$$\langle \widetilde{\mu}_t^N, (1,l) \rangle = \langle \widetilde{\mu}_0^N, (1,l) \rangle,$$
$$\langle \widetilde{\mu}_t^{N,S}, 1 \rangle = \langle \widetilde{\mu}_0^{N,S}, 1 \rangle.$$

- *Moreover,*

$$\sup_{0 \leq t \leq T} \langle \widetilde{\mu}_t^N, 1 \rangle \leq C_0. \tag{42}$$

Proof. As constructed in Theorem 2.2, μ_t^N is obtained from μ_0^N by movement of the monomers, and a sequence of polymerisation or depolymerisation reactions. If $s > 0$ is the time of a polymerisation reaction, we have, for some length $l > 0$ and some location $y \in \mathbb{T}$,

$$\mu_s^N = \mu_{s-}^N - \delta_{(M,y,0)} - \delta_{(S,0,l)} + \delta_{(S,0,l+\frac{1}{N''})}, \tag{43}$$

so after normalization

$$\widetilde{\mu}_s^N = \widetilde{\mu}_{s-}^N - \frac{1}{N}\delta_{(M,y,0)} - \frac{1}{N'}\delta_{(S,0,l)} + \frac{1}{N'}\delta_{(S,0,l+\frac{1}{N''})}. \tag{44}$$

Since $N = N'N''$ we obtain

$$\langle \widetilde{\mu}_s^N, (1,l)\rangle = \langle \widetilde{\mu}_{s-}^N, (1,l)\rangle - \frac{1}{N} - \frac{l}{N'} + \frac{l+\frac{1}{N''}}{N'} = \langle \widetilde{\mu}_{s-}^N, (1,l)\rangle. \tag{45}$$

Similarly we obtain $\langle \widetilde{\mu}_s^{N,S}, 1\rangle = \langle \widetilde{\mu}_{s-}^{N,S}, 1\rangle$, which is only a way to express that the number of polymers is constant. The same cancellations happens for depolymerisation reactions, and we obtain the first item of the Lemma.

Inequality (42) is a consequence of the first two equalities and of (H1) since

$$\langle \widetilde{\mu}_t^N, 1\rangle \leq \langle \widetilde{\mu}_t^N, (1,l)\rangle + \langle \widetilde{\mu}_t^{N,S}, 1\rangle = \langle \widetilde{\mu}_0^N, (1,l)\rangle + \langle \widetilde{\mu}_0^{N,S}, 1\rangle \leq C_0.$$

□

Lemma 2.10. *The sequence $\widetilde{\mu}^N$ is tight in $\mathbb{D}([0,T], \mathcal{M}(\mathcal{P}))$.*

Proof. We use the Roelly criterion (see Lemma A.10, from [9]), which reduces the problem to checking the tightness of the processes $(\langle \widetilde{\mu}^N, f\rangle)_N$ in $\mathbb{D}([0,T],\mathbb{R})$, for every $f \in \mathcal{C}^2(\mathbb{T}) \times \mathcal{C}_b^2(\mathbb{R}_+)$.

We write $\langle \widetilde{\mu}_t^N, f\rangle = V_t^{N,f} + Z_t^{N,f}$ as in (37), and we use the Aldous-Rebolledo criterion (see Lemma A.12, or Theorem 16.9 and 16.10 in [2]): It is enough to show that

1. For every $t > 0$, the sequences $(V_t^{N,f})_N$ and $(\langle Z^{N,f}\rangle_t)_{N\geq 1}$ are tight in $\mathbb{R}$.
2. For any sequence of stopping times $(\tau_N)_{N\geq 1}$ such that $0 \leq \tau_N \leq T$, for every $\varepsilon > 0$, there exists $\eta > 0$ and $N_0 \in \mathbb{Z}_+$ such that

$$\sup_{N\geq N_0} \sup_{\theta\in[0,\eta]} \mathbb{P}(|V_{\tau_N+\theta}^{N,f} - V_{\tau_N}^{N,f}| > \varepsilon) \leq \varepsilon, \tag{46}$$

$$\sup_{N\geq N_0} \sup_{\theta\in[0,\eta]} \mathbb{P}(|\langle Z^{N,f}\rangle_{\tau_N+\theta} - \langle Z^{N,f}\rangle_{\tau_N}| > \varepsilon) \leq \varepsilon. \tag{47}$$

For the first point, we use (38) and then the bound (42) from Lemma 2.9 to obtain the bound

$$\langle Z^{N,f}\rangle_t \leq \frac{C}{N} \sup_{t\in[0,T]} (\langle \widetilde{\mu}_t^N, 1\rangle + \langle \widetilde{\mu}_t^N, 1\rangle^2) \leq \frac{C'}{N},$$

where C depends on $T, D, \lambda^{\pm}, f$, and $C' = C(C_0 + C_0^2)$. The variables $\langle Z^{N,f}\rangle_t$ for $N \in \mathbb{Z}_+^*$ are uniformly bounded so $(\langle Z^{N,f}\rangle_t)_N$ is tight. Similarly, for $t \in [0,T]$, using the the bound (42),

$$V_t^{N,f} \leq C \sup_{t\in[0,T]} (\langle \widetilde{\mu}_t^N, 1\rangle + \langle \widetilde{\mu}_t^N, 1\rangle^2) \leq C'$$

where C depends on $T, D, \lambda^{\pm}, f$, and $C' = C(C_0 + C_0^2)$. So $(V_t^{N,f})_N$ is also tight, and we have the first part of the Aldous-Rebolledo criterion.

Similarly, for any $0 \leq s \leq t \leq T$, we use again (38) followed by the bound (42) from Lemma 2.9 to obtain

$$|\langle Z^{N,f}\rangle_t - \langle Z^{N,f}\rangle_s| \leq \frac{C}{N}(t-s) \sup_{t\in[0,T]} (\langle \widetilde{\mu}_t^N, 1\rangle + \langle \widetilde{\mu}_t^N, 1\rangle^2) \leq \frac{C'}{N}(t-s).$$

where C depends on $T, D, \lambda^{\pm}, f$, and $C' = C(C_0 + C_0^2)$. Therefore for any $N \geq 1$ and any stopping time $0 \leq \tau_N \leq T$, if $0 < \theta < \frac{\varepsilon}{C'}$ we have a.s.

$$|\langle Z^{N,f}\rangle_{\tau_N} - \langle Z^{N,f}\rangle_{\tau_N+\theta}| < \frac{\varepsilon}{N} \leq \varepsilon.$$

It follows that if $0<\eta<\frac{\varepsilon}{C'}$,

$$\sup_{\theta\in[0,\eta]}\mathbb{P}\Big(|\langle Z^{N,f}\rangle_{\tau_N}-\langle Z^{N,f}\rangle_{\tau_N+\theta}|\geq\varepsilon\Big)=0$$

Therefore we obtain (46). Finally, (47) is obtained in exactly the same way. □

Lemma 2.11. *Let ν be any limit of a subsequence of $\widetilde{\mu}^N$ in $\mathbb{D}([0,T],\mathcal{M}(\mathcal{P}))$. Then, ν is deterministic, belongs to $\mathcal{C}([0,T],\mathcal{M}(\mathcal{P}))$, and satifies* (40).

Assume that μ^N (or any of its subsequence, still written μ^N for convenience) converges in distribution in $\mathbb{D}([0,T],\mathcal{M}(\mathcal{P}))$ to $\nu\in\mathbb{D}([0,T],\mathcal{M}(\mathcal{P}))$ for the Skorokhod topology. As the maximum jumps of $\sup_{||f||_\infty\leq1}\langle\mu^N,f\rangle$ are a.s. of size $\leq 1/N'$, we have a.s. $\nu\in\mathcal{C}([0,T],\mathcal{M}(\mathcal{P}))$ (see item (iii) in Lemma A.7).

Define, for $\xi\in\mathbb{D}([0,T],\mathcal{M}(\mathcal{P}))$, $t\geq0$ and $f\in\mathcal{C}^2(\mathbb{T})\times\mathcal{C}^2_b(\mathbb{R}_+)$,

$$\begin{aligned}
\Psi^{N,M}_{t,f}(\xi)&:=\langle\xi^M_t,f^M\rangle-\langle\xi^M_0,f^M\rangle\\
&\quad-\int_0^t\Big(D\langle\xi^M_s,\Delta_yf^M\rangle+\lambda^-\int_0^\infty d\xi^S_s(l)\mathbb{1}_{l\geq\frac{1}{N''}}f^M(0)\\
&\quad-\lambda^+\int_0^\infty\int_{\mathbb{T}}d\xi^S_s(l)d\xi^M_s(y)\mathbb{1}_{l\geq\frac{1}{N''}}\Gamma_\varepsilon(y)f^M(y)\Big)ds,\\
\Psi^{N,S}_{t,f}(\xi)&:=\langle\xi^S_t,f^S\rangle-\langle\xi^S_0,f^S\rangle\\
&\quad-\int_0^t\Big(-\lambda^-\int_0^\infty d\xi^S_s(l)\mathbb{1}_{l\geq\frac{1}{N''}}N''(f^S(l)-f^S(l-\frac{1}{N''}))\\
&\qquad+\lambda^+\int_0^\infty\int_{\mathbb{T}}d\xi^S_s(l)d\xi^M_s(y)\Gamma_\varepsilon(y)\mathbb{1}_{l\geq\frac{1}{N''}}N''(f^S(l+\frac{1}{N''})-f^S(l))\Big)ds,\\
\Psi^{M}_{t,f}(\xi)&:=\langle\xi^M_t,f^M\rangle-\langle\xi^M_0,f^M\rangle\\
&\quad-\int_0^t\Big(D\langle\xi^M_s,\Delta_yf^M\rangle+\lambda^-\int_0^\infty d\xi^S_s(l)\mathbb{1}_{l>0}f^M(0)-\lambda^+\int_0^\infty\int_{\mathbb{T}}d\xi^S_s(l)d\xi^M_s(y)\mathbb{1}_{l>0}\Gamma_\varepsilon(y)f^M(y)\Big)ds,\\
\Psi^{S}_{t,f}(\xi)&:=\langle\xi^S_t,f^S\rangle-\langle\xi^S_0,f^S\rangle\\
&\quad-\int_0^t\Big(-\lambda^-\int_0^\infty d\xi^S_s(l)\mathbb{1}_{l>0}\partial_lf^S(l)+\lambda^+\int_0^\infty\int_{\mathbb{T}}d\xi^S_s(l)d\xi^M_s(y)\Gamma_\varepsilon(y)\mathbb{1}_{l>0}\partial_lf^S(l)\Big)ds,\\
\Psi^{N}_{t,f}(\xi)&:=\Psi^{N,M}_{t,f}(\xi)+\Psi^{N,S}_{t,f}(\xi),\\
\Psi_{t,f}(\xi)&:=\Psi^{M}_{t,f}(\xi)+\Psi^{S}_{t,f}(\xi).
\end{aligned}$$

The goal is to show that $\Psi_{t,f}(\nu)=0$. By summing equations (35) and (36), we obtain $\Psi^N_{t,f}(\widetilde{\mu}^N)=Z^{N,f^M}_t+Z^{N,f^S}_t=Z^{N,f}_t$. It follows from (38) that $\Psi^N_{t,f}(\widetilde{\mu}^N)$ is a martingale of vanishing quadratic variation in the limit $N\to\infty$. Therefore

$$\mathbb{E}[(\Psi^N_{t,f}(\widetilde{\mu}^N))^2]\to0. \tag{48}$$

Then, we have for any ξ,

$$\Psi^M_{t,f}(\xi)-\Psi^{N,M}_{t,f}(\xi)=\int_0^t\Big(-\lambda^-\int_0^\infty d\xi^S_s(l)\mathbb{1}_{0<l<\frac{1}{N''}}f^M(0)+\lambda^+\int_0^\infty\int_{\mathbb{T}}d\xi^S_s(l)d\xi^M_s(y)\mathbb{1}_{0<l<\frac{1}{N''}}\Gamma_\varepsilon(y)\partial_lf^S(l)\Big)ds, \tag{49}$$

so $\Big|\Psi^M_{t,f}(\xi)-\Psi^{N,M}_{t,f}(\xi)\Big|\leq C\sup_{s\leq t}(1+\langle\xi^M_s,1\rangle)\langle\xi^S_s,\mathbb{1}_{0<l<\frac{1}{N''}}\rangle$, where the constant C depends on $t,\lambda^\pm$ and

f. Moreover,

$$\begin{aligned}\Psi^S_{t,f}(\xi)-\Psi^{N,S}_{t,f}(\xi)=&\int_0^t\Big(\lambda^-\int_0^\infty d\xi^S_s(l)\mathbb{1}_{0<l<\frac{1}{N''}}\partial_l f^S(l)\\&-\lambda^-\int_0^\infty d\xi^S_s(l)\mathbb{1}_{l\geq\frac{1}{N''}}\Big(\partial_l f^S(l)-N''(f^S(l)-f^S(l-\frac{1}{N''}))\Big)\\&-\lambda^+\int_0^\infty\int_{\mathbb{T}}d\xi^S_s(l)d\xi^M_s(y)\mathbb{1}_{0<l<\frac{1}{N''}}\Gamma_\varepsilon(y)\partial_l f^S(l)\\&+\lambda^+\int_0^\infty\int_{\mathbb{T}}d\xi^S_s(l)d\xi^M_s(y)\mathbb{1}_{l\geq\frac{1}{N''}}\Gamma_\varepsilon(y)\Big(\partial_l f^S(l)-N''(f^S(l+\frac{1}{N''})-f^S(l))\Big)\Big)ds,\end{aligned}$$

so

$$\Big|\Psi^S_{t,f}(\xi)-\Psi^{N,S}_{t,f}(\xi)\Big|\leq C\sup_{s\leq t}(1+\langle\xi^M_s,1\rangle)\langle\xi^S_s,\mathbb{1}_{0<l<\frac{1}{N''}}\rangle+C\sup_{s\leq t}(1+\langle\xi^M_s,1\rangle)\langle\xi^S_s,1\rangle\frac{1}{N''},$$

where C depends on $t,\lambda^\pm,\Gamma_\varepsilon,f$, and using the assumption that $\partial_l f$ and $\partial^2_l f$ are bounded. Combining the last two bounds, we arrive at

$$\Big|\Psi_{t,f}(\xi)-\Psi^N_{t,f}(\xi)\Big|\leq C\sup_{s\leq t}\Big((1+\langle\xi^M_s,1\rangle)\langle\xi^S_s,\mathbb{1}_{0<l<\frac{1}{N''}}\rangle\Big)+C\sup_{s\leq t}\Big((1+\langle\xi^M_s,1\rangle)\langle\xi^S_s,1\rangle\frac{1}{N''}\Big).\tag{50}$$

Moreover, for $\xi,\eta\in\mathbb{D}([0,T],\mathcal{M}(\mathcal{P}))$, we have

$$\begin{aligned}|\Psi_{t,f}(\xi)-\Psi_{t,f}(\eta)|\leq&|\langle\xi_t-\eta_t,f\rangle|+|\langle\xi_0-\eta_0,f\rangle|\\&+\int_0^t D|\langle\xi^M_s-\eta^M_s,\Delta f^M\rangle|ds\\&+\int_0^t\lambda^-|\langle\xi^S_s-\eta^S_s,1\rangle|(||f^M||_\infty+||\partial_l f^S||_\infty)ds\\&+\int_0^t\lambda^+\Big|\langle\xi^S_s1\rangle\langle\xi^M_s1\rangle-\langle\eta^S_s1\rangle\langle\eta^M_s1\rangle\Big|||\Gamma_\varepsilon||_\infty(||f^M||_\infty+||\partial_l f^S||_\infty)ds\\\leq C|\langle\xi_t-\eta_t,1\rangle|+C|\langle\xi_0-\eta_0,1\rangle|+C\Big(1+\sup_{s\leq t}\langle\xi_s+\eta_s,1\rangle\Big)\int_0^t\lambda^-|\langle\xi_s-\eta_s,1\rangle|ds.\end{aligned}$$

Now, in order to show that $\Psi_{t,f}(\nu)=0$, it is enough to show that

$$\begin{aligned}&\mathbb{E}[|\Psi^N_{t,f}(\widetilde{\mu}^N)|]\to0,\\&\mathbb{E}[|\Psi^N_{t,f}(\widetilde{\mu}^N)-\Psi_{t,f}(\widetilde{\mu}^N)|]\to0,\\&\mathbb{E}[|\Psi_{t,f}(\widetilde{\mu}^N)-\Psi_{t,f}(\nu)|]\to0.\end{aligned}$$

Since $\mathbb{D}([0,T],\mathcal{M}(\mathcal{P}))$ is a Polish space, we can use Skorokhod's representation Theorem, we deduce that it is enough to show the last three bounds when $\widetilde{\mu}^N\overset{a.s.}{\to}\nu$. We assume this almost sure convergence from now on. Going back to the three bounds to show, the first point is a consequence of (48). The second one follows from the following (to be shown) estimates

$$\mathbb{E}[\sup_{s\leq t}\Big((1+\langle\widetilde{\mu}^{N,M}_s,1\rangle)\langle\widetilde{\mu}^{N,S}_s,\mathbb{1}_{0<l<\frac{1}{N''}}\rangle\Big)]\to0,\tag{51}$$

$$\frac{1}{N''}\mathbb{E}[\sup_{s\leq t}\Big((1+\langle\widetilde{\mu}^{N,M}_s,1\rangle)\langle\widetilde{\mu}^{N,S}_s,1\rangle\Big)]\to0,\tag{52}$$

and for the third point, it is enough to show that

$$\mathbb{E}[|\langle\widetilde{\mu}^N_0-\nu_0,1\rangle|]\to0,\tag{53}$$

$$\mathbb{E}[|\langle\widetilde{\mu}^N_t-\nu_t,1\rangle|]\to0,\tag{54}$$

$$\mathbb{E}\Big[\Big(1+\sup_{s\leq t}\langle\widetilde{\mu}^N_s+\nu_s,1\rangle\Big)\int_0^t|\langle\widetilde{\mu}^N_s-\nu_s,1\rangle|ds\Big]\to0.\tag{55}$$

So only these last five convergence results remains to be shown.

According to $(H3)$, the support of the length coordinate of $\widetilde{\mu}_0^{N,S}$ is contained in $\frac{1}{N''}\mathbb{Z}_+$, and that remains true during the whole evolution by construction. We obtain that a.s., for all $s \geq 0$, $\langle \widetilde{\mu}_s^{N,S}, \mathbb{1}_{0<l<\frac{1}{N''}} \rangle = 0$, and the first bound (51) is proven.

Then, using Lemma 2.9, we obtain

$$\sup_{s\leq t}\Big((1+\langle \widetilde{\mu}_s^{N,M},1\rangle)\langle \widetilde{\mu}_s^{N,S},1\rangle\Big) \leq (1+C_0)C_0,$$

which implies the convergence in (52).

Since $\widetilde{\mu}^N \overset{a.s.}{\to} \nu$ in $\mathbb{D}([0,T],(\mathcal{M}(\mathcal{P}),w))$, and ν is continuous, we have $|\langle \widetilde{\mu}_0^N - \nu_0, 1\rangle| \overset{a.s.}{\to} 0$ and $|\langle \widetilde{\mu}_t^N - \nu_t, 1\rangle| \overset{a.s.}{\to} 0$ (see Lemma A.7 (iv)). We can bound these quantities by $C_0 + \sup_{0\leq s\leq T}\langle \nu_s, 1\rangle$ by Lemma 2.9. Moreover thanks to Lemma A.7 (ii), we have

$$\sup_{0\leq s\leq T}\langle \nu_s,1\rangle = \lim_{N\to\infty}\sup_{0\leq s\leq T}\langle \widetilde{\mu}_s^N,1\rangle \leq \sup_N \sup_{0\leq s\leq T}\langle \widetilde{\mu}_s^N,1\rangle \leq C_0. \tag{56}$$

Therefore, using the dominated convergence Theorem, we obtain the convergence in (53) and (54).

Finally, from Lemma A.7 (v), we also have $\int_0^t |\langle \widetilde{\mu}_s^N - \nu_s, 1\rangle| ds \overset{a.s.}{\to} 0$. Using the bound (56) and Lemma 2.9, we arrive at $(1+\sup_{s\leq t}\langle \widetilde{\mu}_s^N + \nu_s, 1\rangle)\int_0^t |\langle \widetilde{\mu}_s^N - \nu_s, 1\rangle| ds \overset{a.s.}{\to} 0$, and the dominated convergence Theorem together with the bound

$$\Big(1+\sup_{s\leq t}\langle \widetilde{\mu}_s^N+\nu_s,1\rangle\Big)\int_0^t |\langle \widetilde{\mu}_s^N-\nu_s,1\rangle| ds \leq (1+2C_0)\int_0^t \sup_{0\leq s\leq T}\langle \widetilde{\mu}_s^N+\nu_s,1\rangle ds \leq (1+2C_0)2C_0T$$

allows to show (55).

Remark 2.12. It is probably possible to weaken the assumption (H1) into

(H1') There is a $\delta > 0$ such that $\sup_N \mathbb{E}[\langle \widetilde{\mu}_0^N, (1, 1+l)\rangle^{2+\delta}] < \infty$,

and Lemmas 2.10 and 2.11 should still hold. In the last part of the proof of Lemma 2.11, we should replace the dominated convergence Theorem with the "uniformly integrable dominated converence Theroem".

The missing result to complete the proof of Conjecture 2.5 is the following:

Conjecture 2.13. *For any $T > 0$, there is at most one solution $\nu \in \mathcal{C}([0,T], \mathcal{M}(\mathcal{P}))$ to equation* (40) *with $\nu_0 = \mu_0$.*

This is purely a PDE statement. There are standard proof for uniqueness of solution to equations with a similar diffusion part (the first equation in (40)), but the transport part (the second equation) has a speed that depends on the state of the system, and an absorbing interface at zero. Therefore, the standard proofs do not adapt well, and specific PDE tools are probably needed.

2.4 Analysis of the limiting PDE in the small reaction radius approximation

Recall the PDE (41) from Proposition 2.7:

$$\begin{aligned}
\partial_t\langle \mu_t^M, f^M\rangle &= D\langle \mu_t^M, \Delta_y f^M\rangle - \Big(\lambda^+\langle \mu_t^M, \Gamma_\varepsilon f^M\rangle - \lambda^- f^M(0)\Big)\langle \mu_t^S, \mathbb{1}_{l>0}\rangle,\\
\partial_t\langle \mu_t^S, f^S\rangle &= \Big(\lambda^+\langle \mu_t^M, \Gamma_\varepsilon\rangle - \lambda^-\Big)\langle \mu_t^S, \mathbb{1}_{l>0}\partial_l f^S\rangle
\end{aligned}$$

We start with some unrelated remarks:

Remark 2.14.

1. There are a lot of stable solutions to (41): assume that $\langle \mu_t^S, \mathbb{1}_{l>0}\rangle = c$ is fixed. If μ_∞^M is stationary for the first equation of (41), i.e. if $D\langle \mu_\infty^M, \Delta_y f^M\rangle - \Big(\lambda^+ \langle \mu_\infty^M, \Gamma_\varepsilon f^M\rangle - \lambda^- f^M(0)\Big)c = 0$ for sufficiently smooth f^M, then in particular for $f^M = 1$ we obtain $\lambda^+\langle \mu_\infty^M, \Gamma_\varepsilon f^M\rangle - \lambda^- = 0$. It means that (μ_∞^M, μ^S) is a stationary solution to (41) for any distribution μ^S of lengths that satisfies $\langle \mu_t^S, \mathbb{1}_{l>0}\rangle = c$.

   In other words, it means that the distribution μ_t^S at large t can be very dependent on the initial condition.

2. The stationary solution μ_∞^M mentionned in the previous point is very close to the constant solution $\mu^M(y) = \frac{\lambda^-}{\lambda^+}$, according to the computation (72), which will be presented in the next section.

3. It is easy to generalize the model to reactions rates λ^- and λ^+ that depend on the length l of the polymer.

Next, we assume that the radius of reaction ε is small (in front of the diffusion constant scale, $\sqrt{D}$, and the size of the torus, 1). Replacing Γ_ε by δ_0 in (41) gives

$$\partial_t \langle \mu_t^M, f^M\rangle = D\langle \mu_t^M, \Delta_y f^M\rangle + \Big(\lambda^- - \lambda^+ \mu_t^M(0)\Big)\langle \mu_t^S, \mathbb{1}_{l>0}\rangle f^M(0), \tag{57}$$

$$\partial_t \langle \mu_t^S, f^S\rangle = -\Big(\lambda^- - \lambda^+ \mu_t^M(0)\Big)\langle \mu_t^S, \mathbb{1}_{l>0}\partial_l f^S\rangle, \tag{58}$$

However, it is now not possible to define the right-hand sides for $\mu \in \mathbb{D}([0,T], \mathcal{M}(\mathcal{P}))$. We have to assume that $\mu_t(0)$ is well defined. However, this additionnal assumption is justifiable. Indeed, μ^M is governed by a parabolic equation, so it is reasonable to assume that is it regular in space. In the end of this section, we pay little attention to the precise space in which the solutions live in. We rather look for typical behaviour they should be exhibiting.

Note that $\langle \mu_t^S, 1\rangle$ is constant (using the equation with $f^S = 1$, or seeing it coming from the stochastic model), but not necessarily $\langle \mu_t^S, \mathbb{1}_{l>0}\rangle$. This corresponds to polymers losing all their monomers and becoming "ghosts".

We look for an explicit solution to this system. We look for a solution in which the measures μ^M and μ^S can be written $\mu_t^M(dy) = \mu_t^M(y)dy$, and $\mu_t^S(dl) = \mu_t^S(l)dl + q_t\delta_0$, with a slight abuse of notation.

A first (but flawed) try is to assume that the density functions are (somewhat) regular in space and time. Then we would have

$$\begin{aligned}
\partial_t \mu_t^M &= D\Delta_y \mu_t^M + \Big(\lambda^- - \lambda^+ \mu_t^M(0)\Big)\langle \mu_t^S, \mathbb{1}_{l>0}\rangle \delta_0,\\
\partial_t \mu_t^S &= \Big(\lambda^- - \lambda^+ \mu_t^M(0)\Big)\partial_l \mu_t^S,\\
\partial_t q_t &= \Big(\lambda^- - \lambda^+ \mu_t^M(0)\Big)\mu_t^S(0).
\end{aligned}$$

This is not what we expect. The last equation should rather be $\partial_t q_t = \lambda^- \mu_t^S(0)$, because dead polymers cannot revive. The assumption that $\mu^S(l)$ is regular in l is actually not good: If $\mu_t^S(l)$ is smooth for some t with $\mu_t^S(0) > 0$, and that $v = \lambda^+\mu_t^M(0) - \lambda^- > 0$, then it is expected that at $t+dt$ we have $\mu_{t+dt}^S(l) = 0$ if $l < m(t+dt)$ for some $m(t+dt) = vdt + o(dt)$, but $\mu_{t+dt}^S(l + m(t+dt)) = \mu_t^S(l)$ for $l > 0$, so there is a discontinuity appearing at $m(t+dt) = vdt + o(dt) > 0$.

In a second try, we track this discontinuity. Define functions $m_0 : \mathbb{R}_+ \to \mathbb{R}$ and $m : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$m_0(t) = \int_0^t \Big(\lambda^+ \mu_s^M(0) - \lambda^-\Big)ds, \tag{59}$$

$$m(t) = m_0(t) - \Big(\inf_{0\le s\le t} m_0(s)\Big). \tag{60}$$

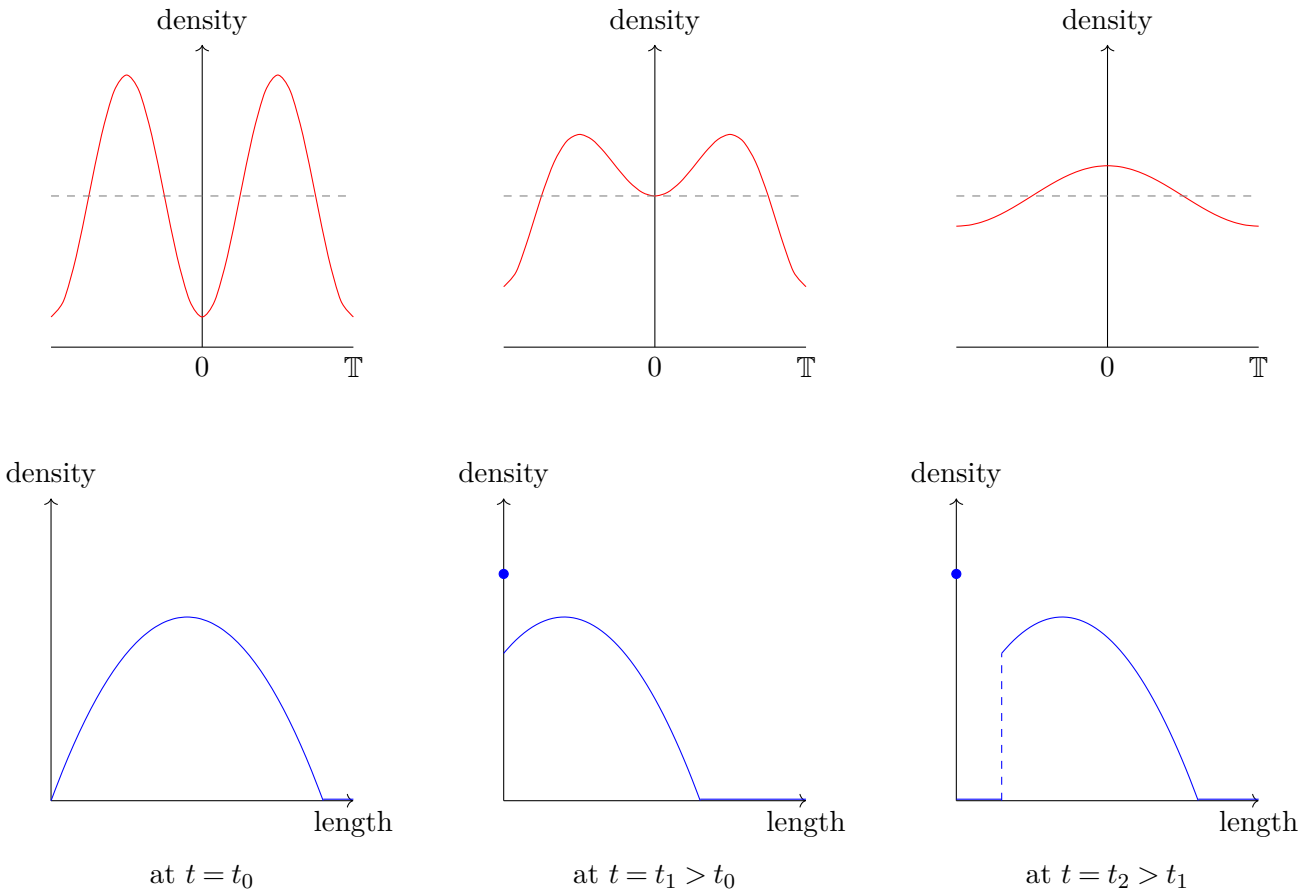

Figure 3: Possible behaviour of the system of PDE given by (41). The densities of μ_t^M and μ_t^S (respectively on the top and bottom lines) have been represented at three times $t_0 < t_1 < t_2$. The blue dot represents the weight of the dirac mass at zero. The horizontal dashed line represents the density of monomers at equilibrium, λ^-/λ^+. At $t = t_0$, the local density of monomers at 0 is below the equilibrium threshold, thus the distribution in lengths of the polymers decreases. The density of monomers at 0 increases due to the polymers depolymerising and the diffusion. At $t = t_1$, the local density of monomers at 0 reaches the critical threshold. From that point, the distribution in lengths of the polymers stops to decrease. Because of the diffusion, the local density of monomers at zero continue to increase after t_1, and goes above the threshold. Thus the distribution in lengths of the polymers increases (except for the ghosts polymers), and reaches the distribution represented at $t = t_2$. Note that the two phases of decrease and increase of the distribution in lengths could continue to alternate indefinetely after t_2, with possible new ghost polymers at zero at each oscillation.

The function m corresponds to the concatenated runs above the running infimum of the function m_0, and will correspond to the point of discontinuity. Note that if $t \mapsto \mu_t^M(0)$ is continuous, then m_0 is differentiable, and for all t, either $m(t) = 0$ and $m_0'(t) = \lambda^+ \mu_t^M(0) - \lambda^- \leq 0$, or m is differentiable at t with $m'(t) = m_0'(t) = \lambda^+ \mu_t^M(0) - \lambda^-$.

Define for $s > 0, y \in \mathbb{T}$,

$$g(s,y) = \frac{1}{\sqrt{4\pi D s}} \sum_{n \in \mathbb{Z}} e^{-\frac{(y+n)^2}{2Ds}}, \tag{61}$$

which is the solution to

$$\partial_t g = D\Delta g \qquad\qquad g(0,\cdot) = \delta_0.$$

We look for a solution to (57) and (58) of the form

$$\mu_t^M(dy) = \Big(\int_0^t \lambda(t-s) g(s,y) ds \Big) dy \tag{62}$$

$$\mu_t^S(dl) = q(t)\delta_0 + \mu_0^s(l - m_0(t)) \mathbb{1}_{l \geq m(t)} dl \tag{63}$$

where $\lambda(t), q(t)$ are unknown, and m_0, m are given above as functions of $\mu_t^M(0) = \int_0^t \lambda(s) g(s,0) ds$.

Substituing into (57) and (58) leads to :

$$\lambda(t) = \Big(\lambda^- - \lambda^+ \mu_t^M(0) \Big) \langle \mu_t^S, \mathbb{1}_{l>0} \rangle.$$

$$\partial_t q(t) f^S(0) = (m - m_0)'(t) \mu_0^S((m-m_0)(t)) f^S(m(t)),$$

where a few lines of informal computations were skipped. From the second equation, we conclude that $\partial_t q(t) = 0$ as soon as $m(t) > 0$. When $m(t) = 0$ we have $m'(t) = 0$ (except on exceptionnal points), so $(m - m_0)'(t) = -m_0'(t) = \lambda^- - \lambda^+ \mu_t^M(0)$, and $\partial_t q(t) = \Big(\lambda^- - \lambda^+ \mu_t^M(0) \Big) \mu_0^S((m-m_0)(t))$. We can summarize the cases in the following formula

$$\partial_t q(t) = \Big(\lambda^- - \lambda^+ \mu_t^M(0) \Big) \mu_0^S((m-m_0)(t)) \mathbb{1}_{m(t)=0}.$$

A more formal way to put it (here the objects do not make complete sense) would be the integrated version

$$q(t) = q(0) + \int_0^t \Big(\lambda^- - \lambda^+ \mu_s^M(0) \Big) \mu_0^S((m-m_0)(s)) \mathbb{1}_{m(s)=0} ds.$$

In total, λ and q are linked by the following set of equations

$$\lambda(t) = \Big(\lambda^- - \lambda^+ \mu_t^M(0) \Big) (\langle \mu_0^S, 1 \rangle + q(0) - q(t)), \tag{64}$$

$$q(t) = q(0) + \int_0^t \Big(\lambda^- - \lambda^+ \mu_s^M(0) \Big) \mu_0^S((m-m_0)(s)) \mathbb{1}_{m(s)=0} ds, \tag{65}$$

where $\mu_t^M(0)$, $m_0(t)$ and $m(t)$ are defined as functions of λ according to (62), (59) and (60). Note that we essentially reduced a set of two coupled PDE into two coupled ODE.

Reciporcally, if λ and q solve (64) and (65), then the functions μ^M and μ^S defined by (62) and (63), should solve (57) and (58).

3 Spatial disparities at equilibrium

The previous model focused on the distribution in lengths of the polymers. However, at equilibrium, the distribution of monomers was homogeneous, at least in the approximation $\Gamma_\varepsilon \to \delta_0$, and we would like to find models in which this is not the case. We now consider more complex models with several types of monomers and polymers, and describe how it influences the spatial distribution of monomers at stationarity. However, we stop to consider the length of every single polymer, and assume that all polymers are always long enough to depolymerise (using the analogy from the previous section, they never become ghosts).

3.1 One type of monomers: No spatial phenomenon

We start with a simple model: there is one type of monomers (M), one type of (simple) polymer (S), and nothing else. We assume that monomers diffuse in $\mathbb{T}$ with diffusion constant $D_M > 0$, and polymers are fixed at 0. We model the dynamic via the following reactions, both localised at zero:

$$S + M \underset{\lambda^-}{\overset{\lambda^+}{\rightleftharpoons}} S. \tag{66}$$

The polymers can be taken out of the equation since they do not evolve. Moreover, it is not necessary to track the number of polymers since it only multiplies the rates λ^- and λ^+ by the same constant. For simplicity, we assume that there is a single polymer. We arrive at a system that is only described by one species "M", and two reactions localised at zero:

$$r^+, r^- : M \underset{\lambda^-}{\overset{\lambda^+}{\rightleftharpoons}} \emptyset. \tag{67}$$

Fix $\lambda^-, \lambda^+, D_M > 0$. Scaling the parameter $\lambda_N^- = N\lambda^-$ and leaving the other parameters constant, we obtain a population of monomers of order of magnitude N. Given a deterministic initial distribution $\mu_{(0)} \in \mathcal{M}(\mathbb{T})$, define $\mu^N \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}(\mathbb{T}))$ to be the associated microscopic dynamic written in notation of Definition 1.2 as $M(\mathbb{T}, \{M\}, \{D_M\}, \{r^+, r^-\}, \emptyset, \{\lambda^+, N\lambda^-\}, \Gamma_\varepsilon, \mu_{(0)})$. The fact that $\tau^\infty = \infty$ follows here simply by stochastic comparison with a pure birth process. Standard arguments, which are simpler versions of the ones presented in Section 2, show that the normalised process $\frac{1}{N}\mu^N$ converges in law to a deterministic measure $\mu \in \mathcal{C}(\mathbb{R}_+, \mathcal{M}(\mathbb{T}))$, that satisfies a weak form of the following PDE:

$$\partial_t \mu_t = D_M \Delta \mu_t + \lambda^- \delta_0 - \lambda^+ \Gamma_\varepsilon \mu_t. \tag{68}$$

We now study the qualitative behaviour of the solution of this PDE. Assuming that ε is small enough, it is reasonable to replace Γ_ε by δ_0, leading to a density satisfying the PDE

$$\partial_t \mu_t = D_M \Delta \mu_t + (\lambda^- - \lambda^+ \mu_t(0))\delta_0. \tag{69}$$

In that case, the stationary distribution μ_∞ satisfies

$$D_M \Delta \mu_\infty = (\lambda^+ \mu_\infty(0) - \lambda^-)\delta_0. \tag{70}$$

The only solution is the constant solution $\mu_\infty(y) = \lambda^-/\lambda^+$. Therefore, we deduce that the localisation of the polymers has no effect on the long-time state of the population of monomers. However, the effect on the out-of-equilibrium states is surely different, but we do not study this question here.

Remark 3.1. We can wonder if choosing a small but positive $\varepsilon > 0$ for Γ_ε will have an effect on the stationary distribution. If $\Gamma_\varepsilon(y) = \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]}$, we can solve explicitely for μ_∞^ε the equation

$$D_M \Delta \mu_\infty^\varepsilon = -\lambda^- \delta_0 + \lambda^+ \Gamma_\varepsilon \mu_\infty^\varepsilon. \tag{71}$$

The solution is

$$\mu_\infty^\varepsilon(x) = \begin{cases} \lambda^- \sqrt{\frac{\varepsilon}{2D_M\lambda^+}} \frac{\cosh\left(\sqrt{\frac{\lambda^+}{2D_M\varepsilon}}(\varepsilon - |x|)\right)}{\sinh\left(\sqrt{\frac{\lambda^+\varepsilon}{2D_M}}\right)} & \text{if } |x| \le \varepsilon \\ \lambda^- \sqrt{\frac{\varepsilon}{2D_M\lambda^+}} \frac{1}{\sinh\left(\sqrt{\frac{\lambda^+\varepsilon}{2D_M}}\right)} & \text{if } |x| > \varepsilon \end{cases} \tag{72}$$

We note that $\mu_\infty^\varepsilon \to \frac{\lambda^-}{\lambda^+}$ uniformly when $\varepsilon \to 0$, so a small positive $\varepsilon > 0$ does not make any significative difference with $\varepsilon = 0$. (see the graph on Desmos).

In Section 4, we ask the question whether it is possible to define a microscopic dynamic μ^N with a reaction radius ε_N that depends on N, such that μ^N converges weakly to a solution of the (69). We will see that it is reasonable to expect this if $N\varepsilon_N \to \infty$.

We also remark the uniform convergence of μ_∞^ε to the constant distribution $\frac{\lambda^-}{\lambda^+}$ in the limit $D_M \to \infty$, for a fixed $\varepsilon > 0$. It corresponds to a very fast diffusion of free monomers.

3.2 Two types of monomers: spatial disparities

We now consider a model with two types of monomers: simple monomers (M), and monomers in a G-actin/profilin complex (P), also called profilactin. We consider one type of (simple) polymer (S). We assume that simple monomers diffuse in $\mathbb{T}$ with constant $D_M > 0$, profilactin complexes diffuse with constant $D_P > 0$, and polymers are fixed at 0. We model the dynamic via the following reactions:

$$S + M \underset{\lambda^-}{\overset{\lambda^+}{\rightleftharpoons}} S \tag{73}$$

$$S + P \overset{\lambda_F^+}{\to} S \tag{74}$$

$$M \overset{\Phi_P}{\to} P. \tag{75}$$

Note that the depolymerisation reaction only releases simple monomers, and not profilactin. The polymers can once again be taken out of the equation, and we are left with the reactions

$$r^+, r^- : M \underset{\lambda^-}{\overset{\lambda^+}{\rightleftharpoons}} \emptyset, \tag{76}$$

$$r_F^+ : P \overset{\lambda_F^+}{\to} \emptyset, \tag{77}$$

$$r_P : M \overset{\Phi_P}{\to} P, \tag{78}$$

where the first two reactions are localised at zero, and the third one non-localised.

Fix $\lambda^-, \lambda^+, \lambda_F^+, D_M, D_P > 0$. Scaling the parameter $\lambda_N^- = N\lambda^-$ and leaving the other parameters constant, we obtain a population of both monomers of order of magnitude N. Given a deterministic initial distribution $\mu_{(0)} \in \mathcal{M}(\{M,P\} \times \mathbb{T})$, define $\mu^N \in \mathbb{D}(R_+, \mathcal{M}(\{M,P\} \times \mathbb{T}))$ to be the associated microscopic dynamic $M(\mathbb{T}, \{M,P\}, \{D_M, D_P\}, \{r^+, r^-, r_F^+\}, \{r_P\}, \{\lambda^+, N\lambda^-, \lambda_F^+, \Phi_P\}, \Gamma_\varepsilon, \mu_{(0)})$ from Definition 1.2. The fact that $\tau^\infty = \infty$ follows here again by stochastic comparison with a pure birth process with birth rate λ^-, since only the reaction r^- creates new monomers.

The normalised microscopic dynamic $\frac{1}{N}\mu^N$ converges again to a deterministic limit $\mu \in \mathcal{C}(\mathbb{R}_+, \mathcal{M}(\{M,P\} \times \mathbb{T}))$. Writing $\mu = (\mu^M, \mu^P)$ with $\mu^M, \mu^P \in \mathcal{C}(\mathbb{R}_+, \mathcal{M}(\mathbb{T}))$, these measures satisfy a weak form of the following PDE:

$$\begin{cases} \partial_t \mu_t^M = D_M \Delta \mu_t^M - \Phi_P \mu_t^M - \lambda^+ \mu_t^M \Gamma_\varepsilon + \lambda^- \delta_0, \\ \partial_t \mu_t^P = D_P \Delta \mu_t^P + \Phi_P \mu_t^M - \lambda_F^+ \mu_t^P \Gamma_\varepsilon. \end{cases} \tag{79}$$

Once again, we assume that ε is small enough so that we can replace Γ_ε with δ_0, and obtain the following system

$$\begin{cases} \partial_t \mu_t^M = D_M \Delta \mu_t^M - \Phi_P \mu_t^M - \lambda^+ \mu_t^M(0) \delta_0 + \lambda^- \delta_0, \\ \partial_t \mu_t^P = D_P \Delta \mu_t^P + \Phi_P \mu_t^M - \lambda_F^+ \mu_t^P(0) \delta_0. \end{cases} \tag{80}$$

A measure $(\mu_\infty^M, \mu_\infty^P) \in \mathcal{C}(\mathbb{R}_+, \mathcal{M}(\{M,P\} \times \mathbb{T}))$ is stationary for the system if it satisfies

$$\begin{cases} 0 = D_M \Delta \mu_\infty^M - \Phi_P \mu_\infty^M - \lambda^+ \mu_\infty^M(0) \delta_0 + \lambda^- \delta_0, \\ 0 = D_P \Delta \mu_\infty^P + \Phi_P \mu_\infty^M - \lambda_F^+ \mu_\infty^P(0) \delta_0. \end{cases} \tag{81}$$

Using Fourier series and Lemma A.14 we find that this stationary distribution satisfies

$$\mu_\infty^M(x) = \frac{\lambda^- - \lambda^+ \mu_\infty^M(0)}{\Phi_P} \frac{A}{\sinh(A)} \cosh(A(2x-1))$$

where

$$A = \sqrt{\frac{\Phi_P}{4D_M}} \tag{82}$$

governs the shape of the distribution. Substituting $x = 0$ we find

$$\mu_\infty^M(0) = \frac{\lambda^-}{\lambda^+} \frac{1}{1 + \frac{\Phi_P}{\lambda^+} \frac{\tanh(A)}{A}}.$$

It is interesting to note that the prefactor $\frac{\lambda^-}{\lambda^+}$ is the equilibrium concentration for simple monomers, when there is no profilactin. In total we arrive at

$$\mu_\infty^M(x) = \frac{\lambda^-}{\lambda^+} \frac{1}{1 + \frac{\Phi_P}{\lambda^+} \frac{\tanh(A)}{A}} \frac{\cosh(A(2x-1))}{\cosh(A)}.$$

Summing the two equations of the system (81) we also have

$$D_M \Delta \mu_\infty^M + D_P \Delta \mu_\infty^P = (\lambda^+ \mu_\infty^M(0) + \lambda_F^+ \mu_\infty^P(0) - \lambda^-)\delta_0.$$

Integrating over $\mathbb{T}$, we find $\lambda^+ \mu_\infty^M(0) + \lambda_F^+ \mu_\infty^P(0) - \lambda^- = 0$ and $D_M \mu_\infty^M + D_P \mu_\infty^P = cste$, so that

$$\begin{aligned}
\mu_\infty^P(x) &= \frac{\lambda^- - \lambda^+ \mu_\infty^M(0)}{\lambda_F^+} - \frac{D_M}{D_P}(\mu_\infty^M(x) - \mu_\infty^M(0)) \\
&= \frac{\lambda^-}{\lambda_F^+} \frac{1}{1 + \frac{\lambda^+}{\Phi_P} \frac{A}{\tanh(A)}} + \frac{D_M}{D_P} \frac{\lambda^-}{\lambda^+} \frac{1}{1 + \frac{\Phi_P}{\lambda^+} \frac{\tanh(A)}{A}} \left(1 - \frac{\cosh(A(2x-1))}{\cosh(A)}\right).
\end{aligned}$$

We remark that:

$$\frac{\mu_\infty^M(1/2)}{\mu_\infty^M(0)} = \cosh(A)$$

only depends on A !

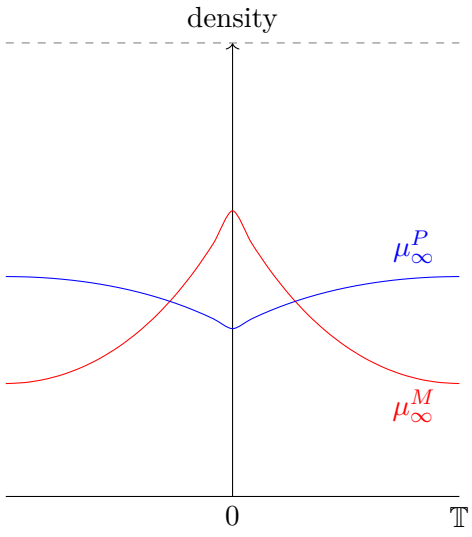

Figure 4: Densities of both types of monomers at equilibrium. This example corresponds to the choice of parameters $\lambda^- = \lambda^+ = \lambda_F^+ = \Phi_P = 1$, $D_M = 0.1$, $D_P = 0.03$, and as a result $A \simeq 1.58$. The dashed line represents the equilibrium density of free monomers M without the "decay" into profilactin complexes.

Note also that if $D_M = D_P$, then the total number of monomers $\mu_\infty^M + \mu_\infty^P$ is constant, and we loose some sort of spatial disparities by stopping to differentiate the species. It makes sense when comparing with Section 3.1.

Remark 3.2. Increasing the number of polymers at the origin amounts to multiplying the reaction constants $\lambda^-, \lambda^+, \lambda_F^+$ by a common factor K. We denote by $\lambda_{(F)}^{K,\pm} = K\lambda_{(F)}^{\pm}$ the new parameters. We can look at the asymptotic distribution of $\mu_\infty^{K,M}$ and $\mu_\infty^{K,P}$ for the stationary distribution with the new parameters $\lambda_{(F)}^{K,\pm}$. As $K \to \infty$, we have

$$\mu_\infty^{K,M}(x) \to \frac{\lambda^-}{\lambda^+}\frac{\cosh(A(2x-1))}{\cosh(A)},$$
$$\mu_\infty^{K,P}(x) \to \frac{D_M}{D_P}\frac{\lambda^-}{\lambda^+}\left(1 - \frac{\cosh(A(2x-1))}{\cosh(A)}\right).$$

Note that the parameter λ_F^+ has no effect in the limit.

3.3 Two types of monomers and two types of polymers: PDMP limit

We now define a third model which exhibits again spatial disparities at stationarity. This model is also interesting in two limit regimes of parameters. In the first case, we obtain a deterministic PDE as before, but in the second case, we have convergence of the microscopic system towards a stochastic system, which in this case is a Piecewise Deterministic Markov Process (PDMP).

In this model, there are simple polymers (S) which can polymerise with simple monomers M, and other polymers bound with formin (F) that can polymerise with profilactin complexes P. Once again, simple monomers diffuse in $\mathbb{T}$ with constant $D_M > 0$, profilactin complexes diffuse with constant $D_P > 0$, and polymers are fixed at 0. Moreover, any polymer can change its type with a given rate $\Phi_F^\pm$. Both polymers depolymerise by releasing simple monomers M. More precisely, the reactions are given by

$$r^+, r^- : S + M \underset{\lambda^-}{\overset{\lambda^+}{\rightleftharpoons}} S \tag{83}$$
$$r_F^- : F \overset{\lambda^-}{\to} F + M \tag{84}$$
$$r_F^+ : F + P \overset{\lambda_F^+}{\to} S \tag{85}$$
$$r_P : M \overset{\Phi_P}{\to} P \tag{86}$$
$$r_S^+, r_S^- : S \underset{\Phi_F^-}{\overset{\Phi_F^+}{\rightleftharpoons}} F. \tag{87}$$

Every reaction, except the reaction $M \to P$, is localised at zero. We can tune the parameters in two ways.

Definition 3.3. Define $\mathcal{P} = \{M, P, S, F\} \times \mathbb{T}$. Fix a set of parameters $\lambda^\pm, \lambda_F^+, \Phi_P, \Phi_F^+, \Phi_F^- > 0$.

Given a possibly random initial condition $\mu_{(0)}^N \in \mathcal{M}(\mathcal{P})$, define $\mu^N \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}(\mathcal{P}))$ to be the process

$$M(\mathbb{T}, \{M,P,S,F\}, \{D_M, D_P, 0, 0\}, \{r^+, r^-, r_F^+, r_F^-, r_S^+, r_S^-\}, \{r_P\}, \{\frac{1}{N}\lambda^+, \lambda^-, \frac{1}{N}\lambda^+, \lambda_F^-, \Phi_F^-, \Phi_F^+, \Phi_P\}, \Gamma_\varepsilon, \nu_{(0)}^N),$$

from Definition 1.2.

Define also $\nu^N \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}(\mathcal{P}))$ to be the process

$$M(\mathbb{T}, \{M,P,S,F\}, \{D_M, D_P, 0, 0\}, \{r^+, r^-, r_F^+, r_F^-, r_S^+, r_S^-\}, \{r_P\}, \{\lambda^+, N\lambda^-, \lambda^+, N\lambda_F^-, \Phi_F^-, \Phi_F^+, \Phi_P\}, \Gamma_\varepsilon, \mu_{(0)}^N),$$

from Definition 1.2.

These sets of parameters imply the following property: in both processes μ^N and ν^N, the number of monomers of both types is of order N. However, in μ^N, the number of polymers of both types is of order N, while in ν^N, this number is of order 1. Define the associated normalisations

$$\widetilde{\mu}^N = \frac{1}{N}\mu^{N,M} + \frac{1}{N}\mu^{N,P} + \frac{1}{N}\mu^{N,S} + \frac{1}{N}\mu^{N,F} = \frac{1}{N}\mu^{N,N}, \tag{88}$$

$$\widetilde{\nu}^N = \frac{1}{N}\nu^{N,M} + \frac{1}{N}\nu^{N,P} + \nu^{N,S} + \nu^{N,F}. \tag{89}$$

$$\tag{90}$$

With suitable conditions on the initial distribution, the process $\widetilde{\mu}^N$ converges as $N \to \infty$ to the solution of the PDE

$$\begin{cases} \partial_t \mu_t^M = D_M \Delta \mu_t^M - \Phi_P \mu_t^M + \lambda^- \delta_0 (\mu_t^S(0) + \mu_t^F(0)) - \lambda^+ \mu_t^S(0) \Gamma_\varepsilon \mu_t^M \\ \partial_t \mu_t^P = D_P \Delta \mu_t^P + \Phi_P \mu_t^M - \lambda_F^+ \mu_t^F(0) \Gamma_\varepsilon \mu_t^P \\ \partial_t \mu_t^S(0) = \Phi_F^- \mu_t^F(0) - \Phi_F^+ \mu_t^S(0) \\ \partial_t \mu_t^F(0) = \Phi_F^+ \mu_t^S(0) - \Phi_F^- \mu_t^F(0) \end{cases} \tag{91}$$

The process $\widetilde{\nu}^N$ converges as $N \to \infty$ to the solution of the following PDMP: the continuous part (ν^M, ν^P) follows, between the jumps, the PDE

$$\begin{aligned} \partial_t \nu_t^M &= D_M \Delta \nu_t^M + \lambda^- \delta_0 (\nu_t^S(0) + \nu_t^F(0)) - \lambda^+ \nu_t^S(0) \Gamma_\varepsilon \nu_t^M - \Phi_P \nu_t^M \\ \partial_t \nu_t^P &= D_P \Delta \nu_t^P - \lambda_F^+ \nu_t^F(0) \Gamma_\varepsilon \nu_t^P + \Phi_P \nu_t^M, \end{aligned} \tag{92}$$

and the discrete part $(\nu^S(0), \nu^F(0)) \in \mathbb{Z}_+^2$ jumps from (n^S, n^F) to $(n^S - 1, n^F + 1)$ with rate Φ_F^+ (if $n^S \geq 1$), and jumps from (n^S, n^F) to $(n^S + 1, n^F - 1)$ with rate Φ_F^- (if $n^F \geq 1$). Note that the jump rates are not affected by the evolution of the continuous part.

4 Results on the microscopic model in a simple case

Let us continue with a simpler model, in which we can push the analysis further than the macroscopic description of large populations. In this model, like in the previous section, the length of the polymers is not considered, and all polymers are always long enough to depolymerise.

4.1 Microscopic model and large population limit

Consider two species : monomers M and (simple) polymers S. Monomers evolve freely in the geographical space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, with a diffusion constant $D = 1/2$, so that each particles follows a standard Brownian motion. Polymers, typically a lot heavier, are fixed at a given point in space, chosen to be $0 \in \mathbb{T}$. They are therefore not explicitly included in the model since they do not move. There are two reactions, respectively of polymerisation and depolymerisation, given by

$$r_+ : M \mapsto \emptyset, \tag{93}$$

$$r_- : \emptyset \mapsto M. \tag{94}$$

Both reactions are localised at zero. The polymerisation reaction r_+ happens at rate $\lambda^+ = 1$, and the depolymerisation reaction happens at rate $\lambda^- = N$.

It is not necessary to track the number of polymers since it only multiplies the rate λ^- by a constant. For simplicity, we assume that there is a single polymer.

Definition 4.1. Given the inital condition $\mu_0^N = 0$, define μ^N to be the microscopic process that follows. In the words of Definition 1.2, μ^N is $M(\mathbb{T}, \{M\}, \{1\}, \{r_+, r_-\}, \emptyset, \{1, N\}, \Gamma_\varepsilon, 0)$.

In [7], it is shown that $\frac{1}{N}\mu_t^N$ converges in distribution in $\mathbb{D}([0,T], \mathcal{M}(\mathbb{T}))$ to the unique solution μ in $\mathcal{C}([0,T], \mathcal{M}(\mathbb{T}))$ of the following differential equation: for every sufficiently regular function f, we have

$$\langle \mu_t, f \rangle = \int_0^t \Big(\frac{1}{2} \langle \mu_s, \Delta f \rangle + f(0) - \langle \mu_s, \Gamma_\varepsilon f \rangle \Big) ds.$$

In other words, it is a weak solution to

$$\partial_t \mu_t = \frac{1}{2}\Delta \mu_t + \delta_0 - \Gamma_\varepsilon \mu_t. \tag{95}$$

Choosing $f = 1$ as a test function gives

$$\partial_t \langle \mu_t, 1 \rangle = 1 - \langle \mu_t, \Gamma_\varepsilon \rangle \tag{96}$$

A specificity of this particularly simple model is that particles do not interact between each other. This means that we can study the trajectory of a single particle, and deduce the total behavior of the system by some kind of law of large numbers. This allows us to answer more questions about the microscopic system.

A question we address here is the following: Assuming that particles are emitted at rate N, and that $\Gamma_\varepsilon = \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]}$, what is the law of the number of particles inside $[-\varepsilon, \varepsilon]$ at time t ? We show in Proposition 4.11 that it is Poisson-distributed, with a mean that converges to $2\varepsilon N$ when $t \to \infty$.

As a consequence, we could take a radius ε_N that depends on N and wonder if the microscopic system converges to a density governed by the equation

$$\partial_t \mu_t = \frac{1}{2}\Delta \mu_t + (1 - \mu_t(0))\delta_0. \tag{97}$$

The result of Proposition 4.11 gives a first indication that it will be the case if $N\varepsilon_N \to \infty$.

Remark 4.2. The choice $\Gamma_\varepsilon = \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]}$ is convenient to define a exact reaction radius ε, but it is not smooth, and therefore does not fit the proof of Proposition 1.6 from [7]. However, we believe that this is only a technical issue and that the result remains true in that case.

4.2 A particle getting absorbed

We consider a particle evolving either in a smooth compact subset of $\mathbb{R}^d$, or in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ for $d \geq 1$. In both cases the space will be denoted by E, and we distinguish the origin $0 \in E$ as a particular point. The particle evolves according to standard Brownian motion (normally reflected at the boundary), started at 0, denoted by B_t. Define a smooth function $\Gamma \in \mathcal{C}^\infty(E, \mathbb{R}_+)$, normalised such that $\int_E \Gamma(x)dx = 1$. We let the particle evolve, but randomly remove it at instantaneous rate $\Gamma(B_t)$. We study the survival time, or absorption time of this particle.

Definition 4.3. Let $(B_t)_{t\geq 0}$ be standard Brownian motion started at 0. Let $\mathcal{E}$ be an exponential variable of parameter 1 independent of B. Define

$$\tau_\Gamma := \inf\left\{ t > 0 : \int_0^t \Gamma(B_s)ds > \mathcal{E} \right\}.$$

Remark 4.4. If we abusively take $\Gamma = \delta_{x_0}$ for some $x_0 \in E$, then $\tau_\Gamma = \inf\{t \geq 0 \mid L_t^{x_0}(B) \geq \mathcal{E}\}$, where $L_t^{x_0}(B)$ is the local time of Brownian motion at point x_0 and time t.

We will show the following result about the law of τ_Γ:

Theorem 4.5. *Let $(\mu_t)_{t\geq 0} \in \mathcal{C}(\mathbb{R}_+, \mathcal{M}(E))$ be the (unique) weak solution to*

$$\partial_t \mu_t = \frac{1}{2}\Delta \mu_t + \delta_0 - \Gamma \mu_t,$$

with $\mu_0 = 0$. Then for $t \geq 0$,

$$\mathbb{P}(\tau_\Gamma \leq t) = \langle \mu_t, \Gamma \rangle. \tag{98}$$

We now emit particles at a constant rate N at the origin, and look at the resulting pool of particles. In particular, a quantity of interest will be the number of extant particles.

Definition 4.6. Define a measure-valued process $\mu'^N \in \mathbb{D}(\mathbb{R}_+, \mathcal{M}(E))$ by emitting new particles at the origin at rate N, and letting each particle evolve independently according to Brownian motion, until it is randomly removed at rate Γ, i.e. letting it evolve until time τ_Γ. The absorption times of different particles are independent.

Proposition 4.7. *The process μ'^N constructed this way is the same in law as μ^N.*

The proof is straightforward, and so we omit it. We are going to define the total number of particles $\langle \mu_t^N, 1\rangle$ in an alternative, more controlled way.

Proposition 4.8. *Let $\mathbb{P}_{\tau_\Gamma}$ be the law of τ_Γ. Define a measure on $\mathbb{R}_+^2$ by $q^N(ds,dr) = N ds\ \mathbb{P}_{\tau_\Gamma}(dr)$. Let Q^N be a Poisson point process in $\mathbb{R}_+^2$ with intensity q^N. Then $(\langle \mu_t^N, 1\rangle)_{t\geq 0}$ has the same law as*

$$\Big(Q^N\big(\{(s,r)\in\mathbb{R}_+^2 : s\leq t < s+r\}\big)\Big)_{t\geq 0}. \tag{99}$$

Proof. Each point (s,r) of Q^N corresponds to a particle emitted at time s, which will survive for a time r, and will thus die at time $s+r$. □

Since $q^N = q^1 + \cdots + q^1$, we have $Q^N = Q^1_{(1)} + \cdots + Q^1_{(N)}$ where all copies of Q^1 are independent, and the law of large numbers allows us to compute the limit of $(\langle \frac{1}{N}\mu_t^N, 1\rangle)_t$ as N tends to infinity.

Proposition 4.9. *The process $(\langle \frac{1}{N}\mu_t^N, 1\rangle)_{t\geq 0}$ converge in distribution for the topology of uniform convergence on compact time intervals to*

$$\left(\int_0^t \mathbb{P}(\tau_\Gamma > s) ds\right)_{t\geq 0}.$$

Proof. Almost sure convergence for a fixed time t follows from the law of large numbers and the fact that

$$\begin{aligned}
\mathbb{E}[Q^1\big(\{(s,r)\in\mathbb{R}_+^2 : s\leq t < s+r\}\big)] &= q^1\big(\{(s,r)\in\mathbb{R}_+^2 : s\leq t < s+r\}\big) \\
&= \int_0^\infty \int_0^\infty \mathbb{1}_{s\leq t<s+r} ds\ \mathbb{P}_{\tau_\Gamma}(dr) \\
&= \int_0^t \mathbb{P}(\tau_\Gamma > t-s) ds \\
&= \int_0^t \mathbb{P}(\tau_\Gamma > s) ds.
\end{aligned}$$

Since the limit is deterministic, the a.s. convergence still holds for finite-dimensional marginals, i.e. for fixed times $0\leq t_1 < \cdots < t_p$, we have $(\langle \frac{1}{N}\mu_{t_i}^N, 1\rangle)_{1\leq i\leq p} \to (\int_0^{t_i} \mathbb{P}(\tau_\Gamma > s) ds)_{1\leq i\leq p}$ a.s.

To obtain a uniform convergence on compact subsets, we now only need the tightness of the process $(\langle \frac{1}{N}\mu_t^N, 1\rangle)_{t\geq 0}$ on compact time intervals. For a fixed $t>0$ and $\delta>0$, the quantity

$$\sup_{t\leq t'\leq t+\delta} \Big|\langle \mu_{t'}^N, 1\rangle - \langle \mu_t^N, 1\rangle\Big| = \sup_{t\leq t'\leq t+\delta} \Big|Q^N\big(\{(s,r)\in\mathbb{R}_+^2 : s\leq t < s+r\leq t'\}\big) - Q^N\big(\{(s,r)\in\mathbb{R}_+^2 : t < s\leq t' < s+r\}\big)\Big|$$

is stochatically dominated by a Poisson variable of mean

$$\begin{aligned}
&q^N\big(\{(s,r)\in\mathbb{R}_+^2 : s\leq t < s+r\leq t+\delta\}\big) + q^N\big(\{(s,r)\in\mathbb{R}_+^2 : t < s\leq t+\delta\}\big) \\
&\qquad \leq q^N\big(\{(s,r)\in\mathbb{R}_+^2 : t < s+r\leq t+\delta\}\big) + q^N\big(\{(s,r)\in\mathbb{R}_+^2 : t < s\leq t+\delta\}\big) \\
&\qquad = N\int_0^{t+\delta} \int_{\min(t-r,0)}^{t+\delta-r} ds \mathbb{P}_{\tau_\Gamma}(dr) + N\int_0^\infty \int_t^{t+\delta} ds \mathbb{P}_{\tau_\Gamma}(dr) \\
&\qquad \leq 2N\delta \int_0^\infty \mathbb{P}_{\tau_\Gamma}(dr) \\
&\qquad = 2N\delta.
\end{aligned}$$

For $N \geq 1$, let X^N be a Poisson random variable of mean $2N\delta$. If $\varepsilon > 0$ is given, we have

$$\begin{aligned}
\sup_{N\geq 1} \frac{1}{\delta}\mathbb{P}\left(\sup_{t\leq s\leq t+\delta}\left|\frac{1}{N}\langle\mu_s^N,1\rangle - \frac{1}{N}\langle\mu_t^N,1\rangle\right| > \varepsilon\right) &\leq \sup_{N\geq 1}\frac{1}{\delta}\mathbb{P}\Big(X^N > N\varepsilon\Big) \\
&\leq \sup_{N\geq 1}\frac{1}{\delta}\mathbb{P}\Big((X^N - 2N\delta) > N(\varepsilon - 2\delta)\Big) \\
&\leq \sup_{N\geq 1}\frac{1}{\delta}\frac{\mathrm{Var}(X^N)}{(\varepsilon-2\delta)^2N^2} \\
&= \frac{4\delta}{(\varepsilon-2\delta)^2}
\end{aligned}$$

For δ small enough, this quantity is smaller than ε, uniformly in $t \in \mathbb{R}_+$. Therefore the process $(\langle \frac{1}{N}\mu_t^N, 1\rangle)_{t\geq 0}$ is tight on compact time intervals, and the Proposition in proven. □

It follows that

$$\langle \mu_t, 1\rangle = \int_0^t \mathbb{P}(\tau_\Gamma > s)ds. \tag{100}$$

Therefore, from Equation (96), we have

$$\langle \mu_t, \Gamma\rangle = 1 - \mathbb{P}(\tau_\Gamma > t), \tag{101}$$

and we obtain Theorem 4.5.

Letting $t \to \infty$ in equation (100) shows that $\langle \mu_t, 1\rangle \to \langle \mu_\infty, 1\rangle := \mathbb{E}[\tau_\Gamma]$. However, the computation of $\langle \mu_\infty, 1\rangle$ is complicated in general and we were not able to give a general formula.

4.3 Number of particles in a given viscinity of zero

We come back to the model with particles being emitted at zero at rate N, and absorbed at rate 1 with proximity kernel Γ. Assume that $\Gamma = \Gamma_\varepsilon := \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]}$ for some reaction radius ε. We ask the question of the typical number of particles inside the reaction radius $[-\varepsilon,\varepsilon]$ during the dynamic described in Section 4.1. Note that Theorem 4.5 is shown for a smooth Γ, but we expect that it remains true for $\Gamma = \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]}$.

The point of view of Proposition 4.8 can be pushed even further:

Proposition 4.10. *Let $b_{\Gamma_\varepsilon} = (B_t)_{0\leq t\leq \tau_{\Gamma_\varepsilon}}$ be Brownian motion in $\mathbb{T}$ killed at the random time $\tau_{\Gamma_\varepsilon}$, and let $\mathbb{P}_{b_{\Gamma_\varepsilon}}$ be its law. Define $\mathcal{Q}^N$ to be a Poisson point process with intensity $Nds\mathbb{P}_{b_{\Gamma_\varepsilon}}(db)$. For a path b in the support of $\mathbb{P}_{b_{\Gamma_\varepsilon}}$, we write $\tau(b)$ for its length. Writing $\mathcal{Q}^N = \sum_i \delta_{(s^i,b^i)}$, we obtain that μ_t has the same law as*

$$\sum_i \delta_{b^i_{t-s^i}}\mathbb{1}_{s^i\leq t<s^i+\tau(b^i)}. \tag{102}$$

Therefore, the number of particles in $[-\varepsilon,\varepsilon]$ at time t has the same law as

$$\mathcal{Q}^N\Big(\{(s,e) : s \leq t < s+\tau(e), e_{t-s} \in [-\varepsilon,\varepsilon]\}\Big),$$

which is a Poisson variable of parameter

$$\begin{aligned}
(Nds\otimes\mathbb{P}_{b_{\Gamma_\varepsilon}})\Big(\{(s,e) \mid s \leq t < s+\tau(e), e_{t-s}\in[-\varepsilon,\varepsilon]\}\Big) &= N\int_0^t \mathbb{P}_{b_{\Gamma_\varepsilon}}(\tau(b) > s, b(s) \in [-\varepsilon,\varepsilon])ds \\
&= N\int_0^t \mathbb{P}(\tau_{\Gamma_\varepsilon}(B) > s, B_s \in [-\varepsilon,\varepsilon])ds,
\end{aligned}$$

where B is a standard Brownian motion under $\mathbb{P}$. This can actually be more or less explicitly computed, at least in the limit $t \to \infty$.

Proposition 4.11. *For any $0 < \varepsilon < 1/2$, the number of particles in $[-\varepsilon, \varepsilon]$ at time t is Poisson-distributed with mean*

$$N \int_0^t \mathbb{P}(\tau_{\Gamma_\varepsilon}(B) > s, B_s \in [-\varepsilon, \varepsilon])ds = 2\varepsilon N \mathbb{P}(\tau_{\Gamma_\varepsilon} \leq t). \tag{103}$$

In particular, this mean converges as $t \to \infty$ to $2\varepsilon N$.

Proof. **Informal argument.** For a fixed $s > 0$, we have

$$\begin{aligned}
-\partial_s \mathbb{P}(\tau_{\Gamma_\varepsilon} > s) &= \lim_{h \to 0^+} \frac{1}{h}(\mathbb{P}(\tau_{\Gamma_\varepsilon} > s) - \mathbb{P}(\tau_{\Gamma_\varepsilon} > s + h)) \\
&= \lim_{h \to 0^+} \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h) \\
&= \lim_{h \to 0^+} \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, B_s \notin [-\varepsilon, \varepsilon]) + \lim_{h \to 0^+} \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, B_s \in [-\varepsilon, \varepsilon]),
\end{aligned}$$

assuming the existence of the limits. Informally, $\frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, B_s \notin [-\varepsilon, \varepsilon])$ is negligeable when h goes to zero, because if B_s is not in the absorbing region $[-\varepsilon, \varepsilon]$, then there is very small probability that it gets absorbed before time $s + h$. For the second term we have

$$\begin{aligned}
\frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, B_s \in [-\varepsilon, \varepsilon]) &= \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon}, B_s \in [-\varepsilon, \varepsilon])\mathbb{P}(\tau_{\Gamma_\varepsilon} \leq s + h \mid s < \tau_{\Gamma_\varepsilon}, B_s \in [-\varepsilon, \varepsilon]) \\
&\simeq \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon}, B_s \in [-\varepsilon, \varepsilon])\frac{h}{2\varepsilon}.
\end{aligned}$$

Indeed, conditionally on $\{s < \tau_{\Gamma_\varepsilon}, B_s \in [-\varepsilon, \varepsilon]\}$, the event $\tau_{\Gamma_\varepsilon}$ roughly corresponds the fact that an exponential clock of parameter $\frac{1}{2\varepsilon}$ rings before time h. Therefore, we should arrive at

$$-\partial_s \mathbb{P}(\tau_{\Gamma_\varepsilon} > s) = \frac{1}{2\varepsilon}\mathbb{P}(s < \tau_{\Gamma_\varepsilon}, B_s \in [-\varepsilon, \varepsilon]), \tag{104}$$

and it is easy to conclude from here.

Real proof. Let us prove this claim with more than heuristic arguments. Fix $s > 0$. For $0 < \delta < \varepsilon$ and $h > 0$, define

$$A_{h,\delta} := \{\forall u \in [s, s + h], |B_u - B_s| \leq \delta\}. \tag{105}$$

By continuity of Brownian motion, we have $\lim_{h \to 0^+} \mathbb{P}(A_{h,\delta}) = 1$. Moreover, by the Markov property, $A_{h,\delta}$ is independent of $(B_u)_{0 \leq u \leq s}$. Recall that $\mathcal{E}$ is an exponential variable of parameter 1, independent of $(B_t)_{t \geq 0}$. Define, for $t \geq 0$, $\mathcal{E}_t := \int_0^t \Gamma_\varepsilon(B_u)du$. Thus $\{\tau_{\Gamma_\varepsilon} > t\} = \{\mathcal{E} > \mathcal{E}_t\}$. We now fix $0 < \delta < \varepsilon$, and write

$$\begin{aligned}
\frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h) &= \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, A_{h,\delta}^c) + \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, A_{h,\delta}), \\
&= \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, A_{h,\delta}^c) && (106) \\
&\quad + \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, A_{h,\delta}, |B_s| < \varepsilon - \delta), && (107) \\
&\quad + \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, A_{h,\delta}, |B_s| > \varepsilon + \delta) && (108) \\
&\quad + \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s + h, A_{h,\delta}, |B_s| \in [\varepsilon - \delta, \varepsilon + \delta]). && (109)
\end{aligned}$$

We now examine the four terms one by one. Using that $\mathcal{E}_{s+h} - \mathcal{E}_s \leq \frac{h}{2\varepsilon}$ a.s., and that conditionally on

$\mathcal{E} > \mathcal{E}_s$, $\mathcal{E} - \mathcal{E}_s$ is an exponential variable of parameter $\frac{1}{2\varepsilon}$, we obtain

$$\begin{aligned}
\frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s+h, A_{h,\delta}^c) &= \frac{1}{h}\mathbb{P}(\tau_{\Gamma_\varepsilon} \leq s+h, A_{h,\delta}^c \mid \tau_{\Gamma_\varepsilon} > s)\mathbb{P}(\tau_{\Gamma_\varepsilon} > s) \\
&\leq \frac{1}{h}\mathbb{P}(\mathcal{E} \leq \mathcal{E}_{s+h}, A_{h,\delta}^c \mid \mathcal{E} > \mathcal{E}_s) \\
&\leq \frac{1}{h}\mathbb{P}(\mathcal{E} - \mathcal{E}_s \leq \mathcal{E}_{s+h} - \mathcal{E}_s, A_{h,\delta}^c \mid \mathcal{E} > \mathcal{E}_s) \\
&\leq \frac{1}{h}\mathbb{P}(\mathcal{E} - \mathcal{E}_s \leq \frac{h}{2\varepsilon}, A_{h,\delta}^c \mid \mathcal{E} > \mathcal{E}_s) \\
&= \frac{1}{h}\mathbb{P}(\mathcal{E} - \mathcal{E}_s \leq \frac{h}{2\varepsilon} \mid \mathcal{E} > \mathcal{E}_s)\mathbb{P}(A_{h,\delta}^c) \\
&= \frac{1}{h}\left(\frac{h}{2\varepsilon} + o(h)\right)\mathbb{P}(A_{h,\delta}^c) \\
&\underset{h\to 0^+}{\longrightarrow} 0,
\end{aligned}$$

since $\mathbb{P}(A_{h,\delta}^c) \underset{h\to 0^+}{\longrightarrow} 0$. Then,

$$\begin{aligned}
&\frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s+h, A_{h,\delta}, |B_s| < \varepsilon - \delta) \\
&\quad= \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s+h, A_{h,\delta}, |B_s| < \varepsilon - \delta, \forall u \in [s, s+h], |B_u| < \varepsilon) \\
&\quad= \frac{1}{h}\mathbb{P}(\tau_{\Gamma_\varepsilon} \leq s+h \mid \tau_{\Gamma_\varepsilon} > s, A_{h,\delta}, |B_s| < \varepsilon - \delta, \forall u \in [s, s+h], |B_u| < \varepsilon) \\
&\qquad\qquad \cdot \mathbb{P}(\tau_{\Gamma_\varepsilon} > s, A_{h,\delta}, |B_s| < \varepsilon - \delta, \forall u \in [s, s+h], |B_u| < \varepsilon) \\
&\quad= \frac{1}{h}\mathbb{P}(\mathcal{E} \leq \mathcal{E}_{s+h} \mid \mathcal{E} > \mathcal{E}_s, A_{h,\delta}, |B_s| < \varepsilon - \delta, \forall u \in [s, s+h], |B_u| < \varepsilon) \\
&\qquad\qquad \cdot \mathbb{P}(\tau_{\Gamma_\varepsilon} > s, A_{h,\delta}, |B_s| < \varepsilon - \delta, \forall u \in [s, s+h], |B_u| < \varepsilon) \\
&\quad= \frac{1}{h}\mathbb{P}(\mathcal{E} - \mathcal{E}_s \leq \frac{h}{2\varepsilon} \mid \mathcal{E} > \mathcal{E}_s, A_{h,\delta}, |B_s| < \varepsilon - \delta, \forall u \in [s, s+h], |B_u| < \varepsilon) \\
&\qquad\qquad \cdot \mathbb{P}(\tau_{\Gamma_\varepsilon} > s, A_{h,\delta}, |B_s| < \varepsilon - \delta, \forall u \in [s, s+h], |B_u| < \varepsilon) \\
&\quad= \frac{1}{h}\left(\frac{h}{2\varepsilon} + o(h)\right)\mathbb{P}(\tau_{\Gamma_\varepsilon} > s, A_{h,\delta}, |B_s| < \varepsilon - \delta, \forall u \in [s, s+h], |B_u| < \varepsilon) \\
&\quad= \left(\frac{1}{2\varepsilon} + o(1)\right)\mathbb{P}(A_{h,\delta}, \tau_{\Gamma_\varepsilon} > s, |B_s| < \varepsilon - \delta) \\
&\quad= \left(\frac{1}{2\varepsilon} + o(1)\right)\mathbb{P}(A_{h,\delta})\mathbb{P}(\tau_{\Gamma_\varepsilon} > s, |B_s| < \varepsilon - \delta) \\
&\quad\underset{h\to 0^+}{\longrightarrow} \frac{1}{2\varepsilon}\mathbb{P}(\tau_{\Gamma_\varepsilon} > s, |B_s| < \varepsilon - \delta).
\end{aligned}$$

Also,

$$\frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s+h, A_{h,\delta}, |B_s| > \varepsilon + \delta) = \frac{1}{h}\mathbb{P}(s < \tau_{\Gamma_\varepsilon} \leq s+h, A_{h,\delta}, |B_s| > \varepsilon + \delta, \forall u \in [s, s+h], |B_u| > \varepsilon) = 0,$$

since $\forall u \in [s, s+h], |B_u| > \varepsilon$ implies that $\mathcal{E}_{s+h} = \mathcal{E}_s$. For the fourth term, using once again that $\mathcal{E}_{s+h} - \mathcal{E}_s \leq$

$\frac{h}{2\varepsilon}$ a.s. we obtain

$$\begin{aligned}
&\frac{1}{h}\mathbb{P}(s<\tau_{\Gamma_\varepsilon}\le s+h, A_{h,\delta}, |B_s|\in[\varepsilon-\delta,\varepsilon+\delta])\\
&\qquad\le \frac{1}{h}\mathbb{P}(s<\tau_{\Gamma_\varepsilon}\le s+h, |B_s|\in[\varepsilon-\delta,\varepsilon+\delta])\\
&\qquad= \frac{1}{h}\mathbb{P}(\mathcal{E}-\mathcal{E}_s\le \mathcal{E}_{s+h}-\mathcal{E}_s, \mathcal{E}>\mathcal{E}_s, |B_s|\in[\varepsilon-\delta,\varepsilon+\delta])\\
&\qquad= \frac{1}{h}\mathbb{P}(\mathcal{E}-\mathcal{E}_s\le \mathcal{E}_{s+h}-\mathcal{E}_s \mid \ |B_s|\in[\varepsilon-\delta,\varepsilon+\delta], \mathcal{E}>\mathcal{E}_s)\mathbb{P}(|B_s|\in[\varepsilon-\delta,\varepsilon+\delta], \mathcal{E}>\mathcal{E}_s)\\
&\qquad\le \frac{1}{h}\mathbb{P}(\mathcal{E}-\mathcal{E}_s\le \frac{h}{2\varepsilon} \mid \ |B_s|\in[\varepsilon-\delta,\varepsilon+\delta], \mathcal{E}>\mathcal{E}_s)\mathbb{P}(|B_s|\in[\varepsilon-\delta,\varepsilon+\delta])\\
&\qquad= \left(\frac{1}{2\varepsilon}+o(1)\right)\mathbb{P}(|B_s|\in[\varepsilon-\delta,\varepsilon+\delta]).
\end{aligned}$$

Therefore, for any $0<\delta<\varepsilon$ we have

$$\begin{aligned}
\left|\frac{1}{h}\mathbb{P}(s<\tau_{\Gamma_\varepsilon}\le s+h)-\frac{1}{2\varepsilon}\mathbb{P}(s<\tau_{\Gamma_\varepsilon}, B_s\in[-\varepsilon,\varepsilon])\right| &\le o(1)+\left(\frac{1}{2\varepsilon}+o(1)\right)\mathbb{P}(|B_s|\in[\varepsilon-\delta,\varepsilon+\delta])\\
&\quad+\left(\frac{1}{2\varepsilon}+o(1)\right)\mathbb{P}(s<\tau_{\Gamma_\varepsilon}, |B_s|\in[\varepsilon-\delta,\varepsilon[) \qquad (110)
\end{aligned}$$

as $h\to 0^+$. Since $\mathbb{P}(|B_s|\in[\varepsilon-\delta,\varepsilon+\delta])\to 0$ as $\delta\to 0$, all terms on the right-hand side of (110) tend to 0 as $\delta\to 0$ and we can conclude that

$$-\partial_s\mathbb{P}(\tau_{\Gamma_\varepsilon}>s)=\lim_{h\to 0^+}\frac{1}{h}\mathbb{P}(s<\tau_{\Gamma_\varepsilon}\le s+h)=\frac{1}{2\varepsilon}\mathbb{P}(s<\tau_{\Gamma_\varepsilon}, B_s\in[-\varepsilon,\varepsilon]).$$

Finally we obtain

$$\int_0^t \mathbb{P}(\tau_{\Gamma_\varepsilon}>s, B_s\in[-\varepsilon,\varepsilon])ds=-2\varepsilon\int_0^t\partial_s\mathbb{P}(\tau_{\Gamma_\varepsilon}>s)ds=2\varepsilon\mathbb{P}(\tau_{\Gamma_\varepsilon}\le t).$$

□

Perspectives

Sections 2, 3 and 4 are three directions in which more effort could be put.

In Section 2, it would be nice to have a proof of the uniqueness of the solution, i.e. a proof of Conjecture 2.13. This requires tools from the theory of PDE. Also, a further analysis of the PDE could give some interesting insights: How does the inital condition impact the mass of "dead" polymers at large times?

In Section 3, we could study the model with two types of monomers and two types of polymers (in subsection 3.3) in more depth, because it has more biological relevance than the previous ones, as it was used in the thesis of Anne Van-Gorp [11]. We could try to describe the stationary solution for the PMDP (92), in diferent regimes, for example when the jumps times are very sparse, or very dense.

Moreover, a model combining the length parameter of Section 2, and several type of monomers or polymers from Section 3 could be made. It is not clear whether new qualitative behaviours will emerge or not.

Finally, in Section 4, we can ask for a proof that the microscopic dynamic with a radius ε_N depending on the number of particles N, converges to the deterministic solution of a PDE as long as $N\varepsilon_N\to\infty$. We can also try to extend the result of Proposition 4.11 to more models, in particular to model where particles do not evolve independently. It would also be interesting to compare the intriguing result of Theorem

4.5 with other similar results such as the Kolmogorov equation, of the Feyman-Kac formula. We can wonder if Thereom 4.5 holds true in a non-compact domain, or for a non-smooth, or even distributional Γ. Finally, when ε goes to 0, and Γ_ε approaches the Dirac mass at zero δ_0, we could recover results about the local time of Brownian motion.

A Appendix: Mathematical tools

A.1 Stochastic calculus

All the objects defined in this section are supposeed to be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition A.1. 1. A *finite variation process* is an adapted process $(A_t)_{t\geq 0}$ with trajectories that are a.s. càdlàg and with finite variation on every compact interval, i.e., for all $t \geq 0$,

$$\sup_{\substack{p\geq 1 \\ 0=t_0<\cdots<t_p=t}} \sum_{i=1}^{p} |A_{t_i} - A_{t_{i-1}}| < \infty \quad \text{a.s.}$$

2. A *semimartingale* is an adapted process $X = (X_t)_{t\geq 0}$ with càdlàg trajectories, that can be decomposed as
$$X_t = X_0 + M_t + A_t, \tag{111}$$
where X_0 is $\mathcal{F}_0$ measurable, $(M_t)_{t\geq 0}$ is a local martingale, and $(A_t)_{t\geq 0}$ is a finite variation process. Note that the decomposition is not unique. See [8], Theorem 1 page 102, for alternative definitions of a semimartingale.

3. Two adapted processes X, Y are *indistinguishable* if a.s. we have $\forall t \geq 0, X_t = Y_t$.

4. Given a semimartingale X and an adapted càglàd process H, it is possible to define the stochastic integral
$$Y_t = \int_0^t H_s dX_s.$$
The resulting process Y is an adapted càdlàg process (see [8], Chap. II, sec. 4, page 59). We refer to [8], chapter II, for standard results on stochastic integrals. Moreover this integration is a.s. equal to its Riemann integral version when X is a finite variation process.

5. The *quadratic variation* of a semimartingale X is the finite variation process $[X] = ([X]_t)_{t\geq 0}$ defined for $t \geq 0$ by
$$[X]_t = X_t^2 - 2\int_0^t X_{s-} dX_s. \tag{112}$$
Note that $[X]_0 = X_0^2$ (see [8], Chap. II, sec. 6).

6. The quadratic covariation $[X, Y]$ of two semimartingales X, Y is the finite variation process defined for $t \geq 0$ by
$$[X, Y]_t = X_t Y_t - \int_0^t X_{s-} dY_s - \int_0^t Y_{s-} dX_s.$$
The quadratic covariation is bilinear, symetric, and satisfies $[X, X] = [X]$ for any semimartingale X (see [8], chap. II sec. 6).

7. An adapted process X is *locally integrable* if there exists a sequence of stopping times $(T_n)_{n\geq 1}$ with $T_n \to \infty$ a.s., and such that for any $n \geq 1$, $\mathbb{E}[\sup_{0\leq t\leq T_n} |X_t|] < \infty$ (see [8], note at the bottom of page 124).

8. An adapted process X is called *predictable* if it is measurable with respect to the σ-algebra generated by all adapted càglàd processes.

9. Let X be a semimartingale such that $[X]$ is locally integrable. Then, we can define the *predictable quadratic variation* of X to be the only predictable finite variation process $\langle X\rangle = (\langle X\rangle_t)_{t\geq 0}$ such that $([X]_t - \langle X\rangle_t)_{t\geq 0}$ is a martingale (see [8], definition at the end of page 124).

Lemma A.2 (Kunita-Watanabe inequality)**.** *Let X, Y be two semimartingales, and H, K be two adapted càglàd processes. Then alsmost surely,*

$$\left(\int_0^t |H_s K_s||d[X,Y]_s|\right)^2 \leq \left(\int_0^t H_s^2 d[X]_s\right)\left(\int_0^t K_s^2 d[Y]_s\right). \tag{113}$$

Proof. See [8], Theroem 25, page 69. □

Lemma A.3. *Let X be a semimartingale and V be a continuous adapted process (hence càdlàg), with $V_0 = 0$. Then the processes $[X+V]$ and $[X]$ are indistinguishable.*

Proof. Let $t \geq 0$. It follows from standard integration that $[V]_t = V_t^2 - 2\int_0^t V_s dV_s = V_0^2 = 0$. Moreover, the Kunita-Watanabe inequality (113) with $H = K = 1$, $Y = V$ implies that ,$\int_0^t |d[X,V]_s| \leq 0$ so $[X,V]_t = 0$ for Lebesgue almost all $t \geq 0$. Hence $[X+V]_t = [X+V, X+V]_t = [X,X]_t + 2[X,V]_t + [V,V]_t = [X]_t$ for Lebesgue almost all $t \geq 0$. Since both processes $[X+V]$ and $[X]$ are càdlàg and have the same values for Lebesgue almost all times, they are indistinguishable. □

A.2 Skorokhod spaces and tightness criteria

Let (E,d) be a Polish space. We will use the Skorokhod space $\mathbb{D}([0,T],E)$ of all càdlàg paths with values in E, and defined over the time interval $[0,T]$. This space is equipped with the usual Skorokhod topology (also called J1 topology). We first recall the definition of convergence for this topology.

Definition A.4.

- A *time change* is a function $\lambda : [0,T] \to [0,T]$ that is increasing and continuous, with $\lambda(0) = 0, \lambda(T) = T$. Hence a time change is always bijective., with a continuous reverse bijection.
- Let $x^N, x \in \mathbb{D}([0,T],E)$ for every $N \geq 1$. We say that x^N converges to x in $\mathbb{D}([0,T],E)$ if there exist time changes $\lambda^N, N \geq 1$, such that

$$||\lambda^N - \mathrm{Id}_{[0,T]}||_\infty \underset{N\to\infty}{\longrightarrow} 0, \tag{114}$$

$$\sup_{0\leq t\leq T} d(x^N_{\lambda^N(t)}, x_t) \underset{N\to\infty}{\longrightarrow} 0. \tag{115}$$

It is known that any path $x \in \mathbb{D}([0,T],E)$ has a countable number of discontinuities, see e.g. [6] Lemma 5.1 page 116. We continue with standard properties of convergence in Skorokhod spaces over a Polish space.

Lemma A.5. *Let E be a Polish space, $x^N, x \in \mathbb{D}([0,T],E)$ for $N \geq 1$, and assume that $x^N \to x$ in $\mathbb{D}([0,T],E)$. Then*

(i) If F is another Polish space and $f : E \to F$ is continuous, then $(f(x_t^N))_{0\leq t\leq T} \to (f(x_t))_{0\leq t\leq T}$ in $\mathbb{D}([0,T],F)$.

(ii) If $E = \mathbb{R}$ then $\sup_{0\leq t\leq T}|x_t^N| \to \sup_{0\leq t\leq T}|x_t|$.

(iii) If $E = \mathbb{R}$ then $\sup_{0\leq t\leq T}|\Delta x_t^N| \to \sup_{0\leq t\leq T}|\Delta x_t|$.

(iv) Let $t \in [0,T]$. If x is continuous at t then $x_t^N \to x_t$ in E.

(v) Let $f : E \to \mathbb{R}$ be continuous. Then $\int_0^T f(x_t^N)dt \to \int_0^T f(x_t)dt$.

Proof. (i) It is a simpler version of Exercice 13 page 151 of [6].

(ii) There exist time changes $\lambda^N : [0,T] \to [0,T]$ such that $||\lambda^N - \mathrm{Id}\,||_\infty \to 0$ and $||x^N \circ \lambda^N - x||_\infty \to 0$. Thus

$$\Big|||x^N||_\infty - ||x||_\infty\Big| = \Big|||x^N \circ \lambda^N||_\infty - ||x||_\infty\Big| \leq ||x^N \circ \lambda^N - x||_\infty \to 0,$$

where we used the triangle inequality for the infinite norm.

(iii) There exists time changes $\lambda^N : [0,T] \to [0,T]$ such that $||\lambda^N - \mathrm{Id}\,||_\infty \to 0$ and $||x^N \circ \lambda^N - x||_\infty \to 0$. Thus

$$\begin{aligned}\Big|\sup_{0\leq t\leq T}|\Delta x_t^N| - \sup_{0\leq t\leq T}|\Delta x_t|\Big| &= \Big|\sup_{0\leq t\leq T}|\Delta(x^N \circ \lambda^N)_t| - \sup_{0\leq t\leq T}|\Delta x_t|\Big| \\ &\leq \sup_{0\leq t\leq T}|\Delta(x^N \circ \lambda^N - x)_t| \\ &\leq 2||x^N \circ \lambda^N - x||_\infty \to 0.\end{aligned}$$

(iv) We have

$$d(x_t^N, x_t) \leq d(x_t^N, x_{\lambda^N(t)}) + d(x_{\lambda^N(t)}, x_t) \to 0,$$

by condition (114) for the first term, and because $\lambda_t^N \to t$ by (115), and x is continuous at t for the second term.

(v) Let $y_t^N = f(x_t^N)$ and $y_t = f(x_t)$. We have $y^N \to y$ in $\mathbb{D}([0,T],\mathbb{R})$ by point (i). Moreover by point (ii) we also have $||y^N||_\infty \to ||y||_\infty$, so $||y^N||_\infty \leq C$ for some finite constant C. Let $D \subseteq [0,T]$ be the (countable) set of discontinuity points of y. We have by point (iv), for all $t \in [0,T]$, $y_t^N \mathbb{1}_{t\notin D} \to y_t \mathbb{1}_{t\notin D}$. Thefore by dominated convergence,

$$\int_0^T y_t^N \mathbb{1}_{t\notin D} dt \to \int_0^T y_t \mathbb{1}_{t\notin D} dt.$$

Since D is countable, the result follows.

□

Remark A.6. Point (iv) is not necessarily true if x is not continuous at t: see Theorem 12.5(i) in Billingsley [2], p.134.

If not mentionned, the space $\mathcal{M}(E)$ of all finite measures on E is equipped with the topology of *weak convergence*; that is, μ^N converges to μ in $\mathcal{M}(E)$ if for any continuous bounded function $f : E \to \mathbb{R}$ we have $\langle \mu^N, f\rangle \to \langle \mu, f\rangle$. Then the space $\mathcal{M}(E)$ is Polish. But $\mathcal{M}(E)$ can also be equipped with the *vague topology*, for which test functions are supposed to be compactly supported. This second topological space will be written $(\mathcal{M}(E), v)$. Note that $(\mathcal{M}(E), v)$ is in general not a Polish space (see this discussion on StackExchange). In particular, we will use the space $\mathbb{D}([0,T],\mathcal{M}(E))$, with $E \subseteq \mathbb{R}^d \times \mathbb{T}^{d'}$. Let us translate the results of Lemma A.5 on this particular space.

Lemma A.7. *Let E be a Polish space, $\mu^N, \mu \in \mathbb{D}([0,T],\mathcal{M}(E))$ for $N \geq 1$, and assume that $\mu^N \to \mu$ in $\mathbb{D}([0,T],\mathcal{M}(E))$. Then*

(i) For any measurable bounded $f : E \to \mathbb{R}$, we have $\langle \mu^N, f\rangle \to \langle \mu, f\rangle$ in $\mathbb{D}([0,T],\mathbb{R})$.

(ii) We have $\sup_{0\leq t\leq T}\langle \mu_t^N, 1\rangle \to \sup_{0\leq t\leq T}\langle \mu_t, 1\rangle$.

(iii) For any measurable bounded $f : E \to \mathbb{R}$, we have $\sup_{0\leq t\leq T}\Delta\langle \mu_t^N, f\rangle \to \sup_{0\leq t\leq T}\Delta\langle \mu_t, f\rangle$.

(iv) *If $\mu \in \mathcal{C}([0,T],\mathcal{M}(E))$, then for all $0 \le t \le T$, we have $\mu_t^N \to \mu_t$ in $\mathcal{M}(E)$. As a consequence, for any measurable bounded $f : E \to \mathbb{R}$, we have $\langle \mu_t^N, f\rangle \to \langle \mu_t, f\rangle$.*

(v) *We have $\int_0^T \langle \mu_s^N - \mu_s, 1\rangle ds \to 0$.*

Proof. These results follow from Lemma A.5, combined with the fact that for any measurable and bounded $f : E \to \mathbb{R}$, the map $\mu \in \mathcal{M}(E) \mapsto \langle \mu, f\rangle \in \mathbb{R}$ is continuous. □

Lemma A.8. *The space $\mathbb{D}([0,T],\mathcal{M}(E))$ is a Polish space for the Skorokhod topology.*

Proof. We refer to Bilingsley [2], Theorem 12.2 p.128 for a proof that $\mathbb{D}([0,T],\mathbb{R})$ is a Polish space. According to the note at the end of page 121, is remains true (with the same proof) if we replace $\mathbb{R}$ by a Polish space. Since the space $\mathcal{M}(E)$ equipped with the weak topology is Polish, the Lemma follows. □

Remark A.9. The Skorokhod topology is not compatible with the vector space structure. For example, we can find a sequence $(\mu^N)_{N\ge1}$ and μ such that $\mu^N \to \mu$ but $\mu^N - \mu$ does not converge to 0.

We also give criteria for tightness of random variables taking values in a Skorokhod space. We will use two important tools towards this goal. The first one is the reduction to real-valued trajectories, and the second one is a tightness criterion for real-valued trajectories.

Lemma A.10. *(Roelly's criterion) Let $T > 0$, E be a compact Polish space, and $(\mu^N)_{N\ge1}$ be a sequence of $\mathbb{D}([0,T],\mathcal{M}(E))$-valued random variables. Assume that for any bounded continuous $f : E \to \mathbb{R}$ (or a dense subset of it for the uniform topology), the sequence $(\langle \mu^N, f\rangle)_{N\ge1}$ is tight in $\mathbb{D}([0,T],\mathbb{R})$. Then the sequence $(\mu^N)_{N\ge1}$ is tight in $\mathbb{D}([0,T],\mathcal{M}(E))$.*

Remark A.11. If E is not compact, we can use its one-point-compactification $\overline{E} := E \sqcup \{\infty\}$. If for any measurable bounded $f : \overline{E} \to \mathbb{R}$ (or a dense subset of it for the uniform topology), the sequence $(\langle \mu^N, f\rangle)_{N\ge1}$ is tight in $\mathbb{D}([0,T],\mathbb{R})$, then the sequence $(\mu^N)_{N\ge1}$ is tight in $\mathbb{D}([0,T],\mathcal{M}(\overline{E}))$. Moreover, for any limit μ of a converging subsequence, we have $\langle \mu^N, 1\rangle \to \langle \mu, 1\rangle$, so by Proposition 1 in [10], we also have $\mu^N \to \mu$ in $\mathbb{D}([0,T],\mathcal{M}(E))$. Therefore $(\mu^N)_N$ is tight in $\mathbb{D}([0,T],\mathcal{M}(E))$.

Let us also give a criterion for tightness or real-valued semimartingale, using their decomposition into a sum of a local martingale and a finite variation process. We refer to the introduction of [5] for background.

Lemma A.12. *(Aldous-Rebolledo tightness criterion) Let $(Y^N)_{N\ge1}$ be a sequence of $\mathbb{D}([0,T],\mathbb{R})$-valued random variables. Assume that it can be written $Y_t^N = V_t^N + M_t^N$ where for every $N \ge 1$, V^N is a finite variation process, and M^N is a martingale. Assume also that*

1. *For every $t > 0$, the sequences $(V_t^N)_N$ and $(\langle M^N\rangle_t)_{N\ge1}$ are tight in $\mathbb{R}$.*
2. *For any sequence of stopping time $0 \le \tau_N \le T$, for every $\varepsilon > 0$, there exists $\eta > 0$, and $N_0 \in \mathbb{Z}_+$ such that*

$$\sup_{N\ge N_0} \sup_{\theta\in[0,\eta]} \mathbb{P}(|V^N_{\tau_N+\theta} - V^N_{\tau_N}| > \varepsilon) \le \varepsilon, \tag{116}$$

$$\sup_{N\ge N_0} \sup_{\theta\in[0,\eta]} \mathbb{P}(|\langle M^N\rangle_{\tau_N+\theta} - \langle M^N\rangle_{\tau_N}| > \varepsilon) \le \varepsilon. \tag{117}$$

Then the sequence $(Y^N)_{N\ge1}$ is tight.

A.3 Fourier series

We will also use Fourier series for functions on $\mathbb{T}$.

Definition A.13. The Fourier coefficients of $f \in L^2(\mathbb{T})$ are given by $\hat{f}(n) = \int_0^1 f(x)e^{-2i\pi nx}dx$, and f is given back by $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2i\pi nx}$. The Fourier series is expanded to distributions, and for example $\hat{\delta}_0(n) = 1$.

Lemma A.14. *Let $A > 0$, and $f_A(x) = \cosh(A(2x-1))$ for $x \in \mathbb{T} = [0,1[$. Its Fourier series is given by*

$$\hat{f}_A(n) = \frac{\sinh(A)}{A}\frac{1}{\frac{\pi^2 n^2}{A^2}+1}.$$

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